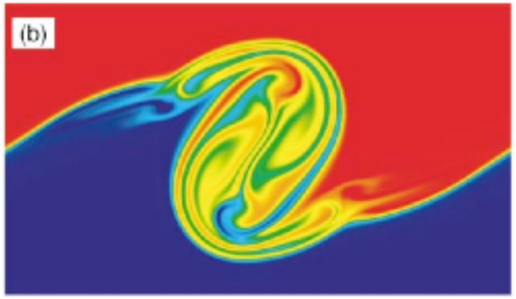
# SIO 210 CSP: Dynamics II
# Momentum balance (no rotation)

L. Talley Fall, 2016

- Continuity (mass conservation) and Fick's Law
  - Reading:
    - » DPO ch. 5.1

- Force balance
  - Reading:
    - » DPO ch. 7.1, 7.2 (skip 7.2.3)




- Lecture emphasis: advection, pressure gradient force, eddy viscosity

# Equations for fluid mechanics (for the ocean)

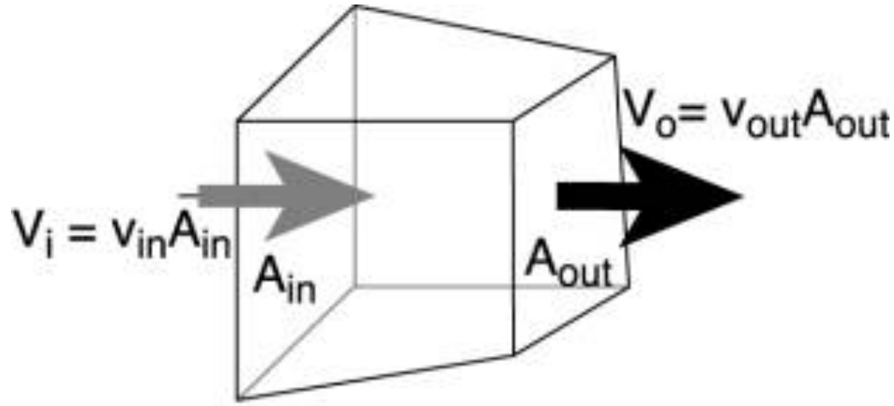
- Mass conservation (continuity) (no holes) (covered in previous lecture)

- Force balance: Newton's Law ( $\vec{F} = m\vec{a}$ ) (3 equations)
- Equation of state (for oceanography, dependence of density on temperature, salinity and pressure) (1 equation)
- Equations for temperature and salinity change in terms of external forcing, or alternatively an equation for density change in terms of external forcing (2 equations)

- 7 equations to govern it all

# Review: Conservation of volume: Continuity at a point

Conservation at a point in the fluid (shrink the box to a point):




- 1D: $0 = \Delta u/\Delta x = \partial u/\partial x$
- 2D: $0 = \Delta u/\Delta x + \Delta v/\Delta y = \partial u/\partial x + \partial v/\partial y$
- 3D: $0 = \Delta u/\Delta x + \Delta v/\Delta y + \Delta w/\Delta z = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z$
- (Net convergence or divergence within the ocean results in mounding or lowering of sea surface, or within isopycnal layers, same thing) NO holes in the ocean

# Force balance in a fluid

- Newton's law

  **F** = m**a** (from physics class)

  This is a vector equation, with 3 equations for each of the three directions (x, y and z)

  m**a** = **F** (for fluids)

- In a continuous fluid

  Divide by volume, so express in terms of density $\rho$ and force per unit volume $\Im$:

  $\rho\mathbf{a} = \Im$

  "Acceleration" in a fluid has two terms: actual acceleration and advection

# Time change and Acceleration

- Time change the change in stuff with time, for instance temperature T or heat Q = $\rho c_p T$:

$$\Delta T/\Delta t \Rightarrow \partial T/\partial t$$

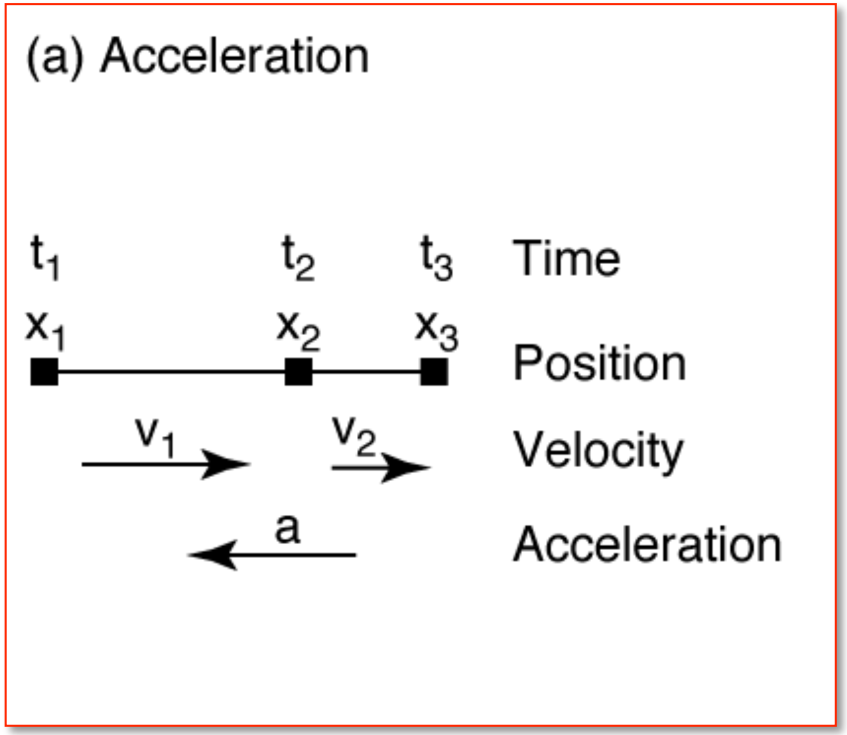
$$\Delta Q/\Delta t \Rightarrow \partial Q/\partial t$$

(Units are stuff/sec; here heat/sec or J/sec or W)

- Acceleration: the change in velocity with time

$$\Delta u/\Delta t \Rightarrow \partial u/\partial t$$

(Units are velocity/sec, hence m/sec$^2$)




DPO Fig. 7.1

# Advection

- Move "stuff" - temperature, salinity, oxygen, momentum, etc.

- By moving stuff, we might change the value of the stuff at the next location. We only change the value though if there is a difference ("gradient") in the stuff from one point to the next

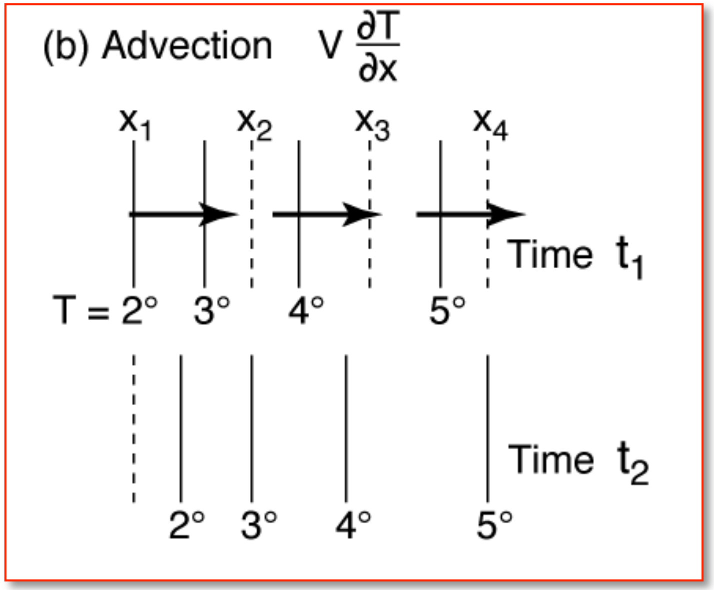
- Advection is proportional to velocity and in the same direction as the velocity
- E.g. $u \, \Delta T/\Delta x$ or $u \, \partial T/\partial x$ is the advection of temperature in the x-direction

- Effect on time change of the property:

$$\partial T/\partial t = -u \, \partial T/\partial x$$

- Advection can act on velocity as well:

$$\partial u/\partial t = -u \, \partial u/\partial x$$



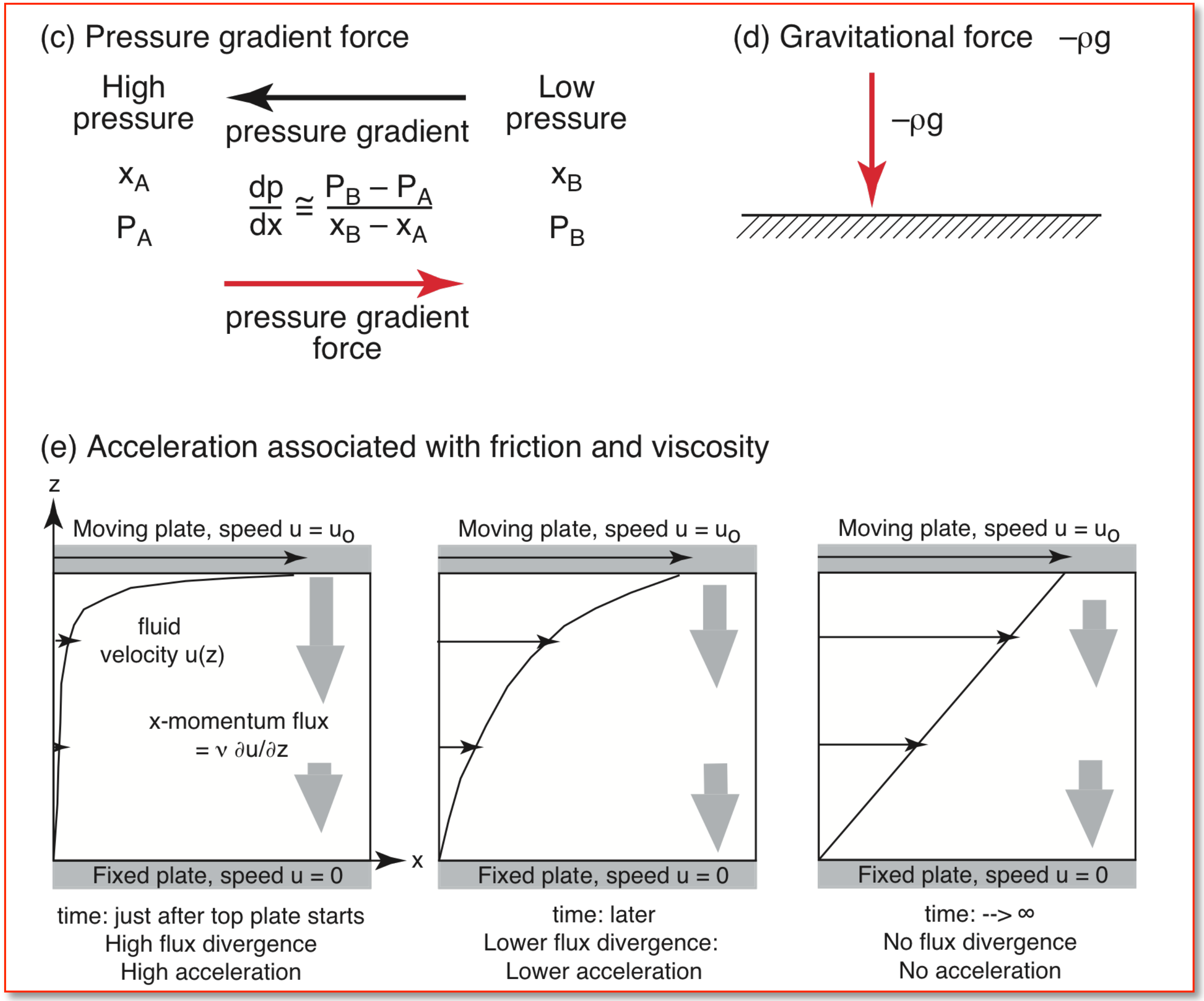

DPO Fig. 7.1

# Forces acting on geophysical fluid

1. Gravity: $g = 9.8\ m/sec^2$

2. Pressure gradient force

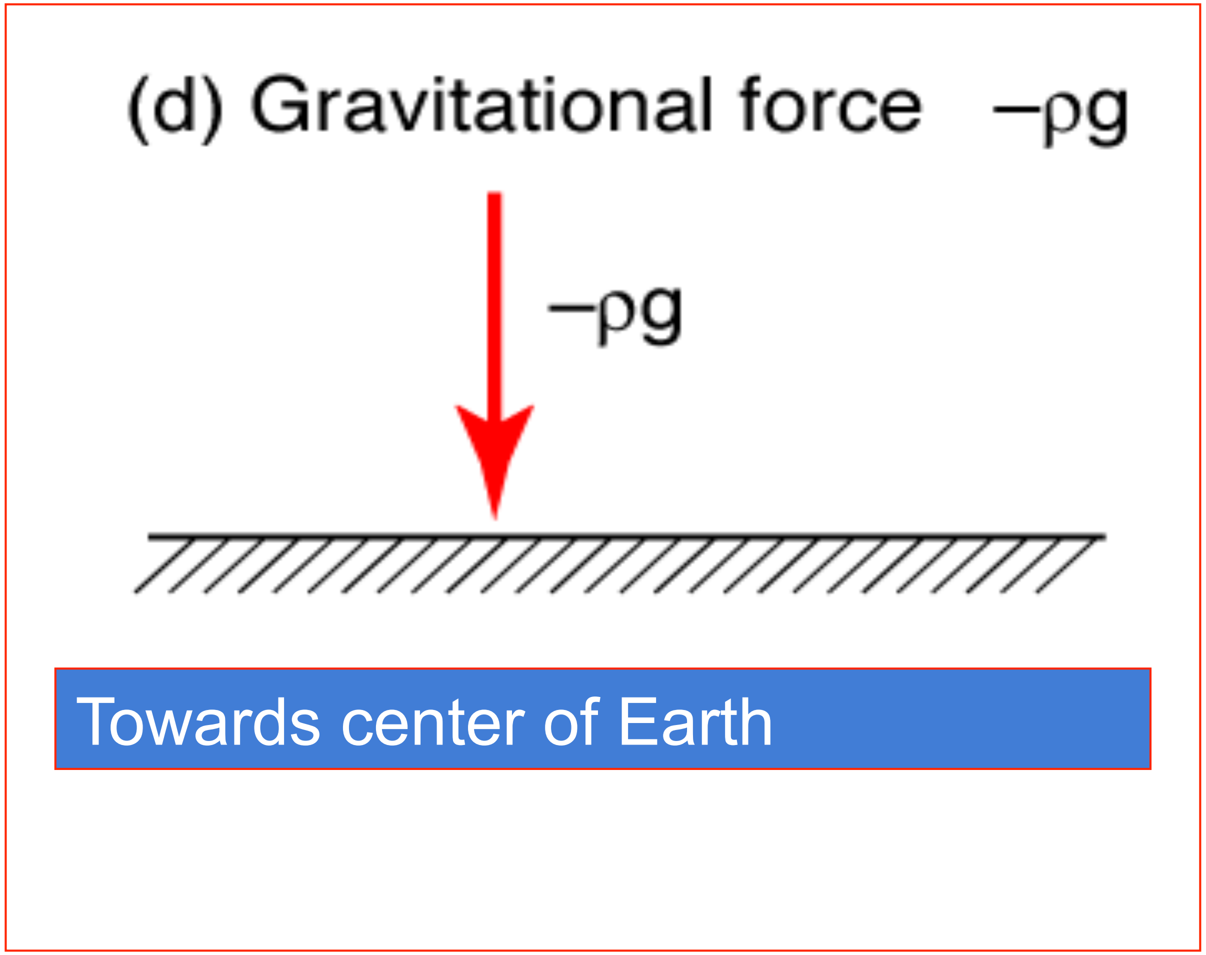
3. Friction (dissipation) (viscous force)




DPO Fig. 7.1

# Forces acting on geophysical fluid

1. Gravity: $g = 9.8\ m/sec^2$
2. Pressure gradient force
3. Friction (dissipation) (viscous force)



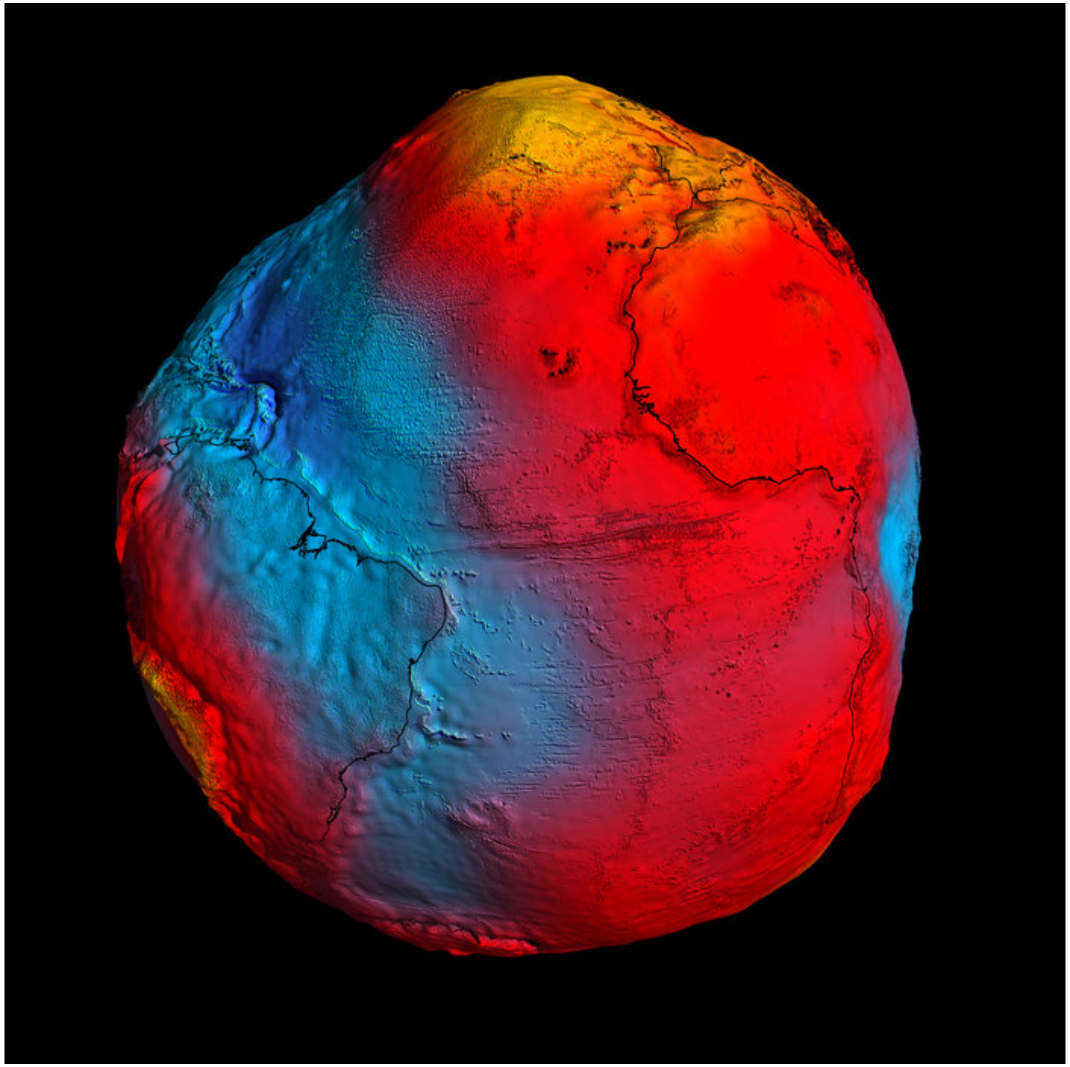

DPO Fig. 7.1

# Forces acting on geophysical fluid

1. Gravity: $g = 9.8\ m/sec^2$

Aside: what is the real shape of the constant gravity surface?

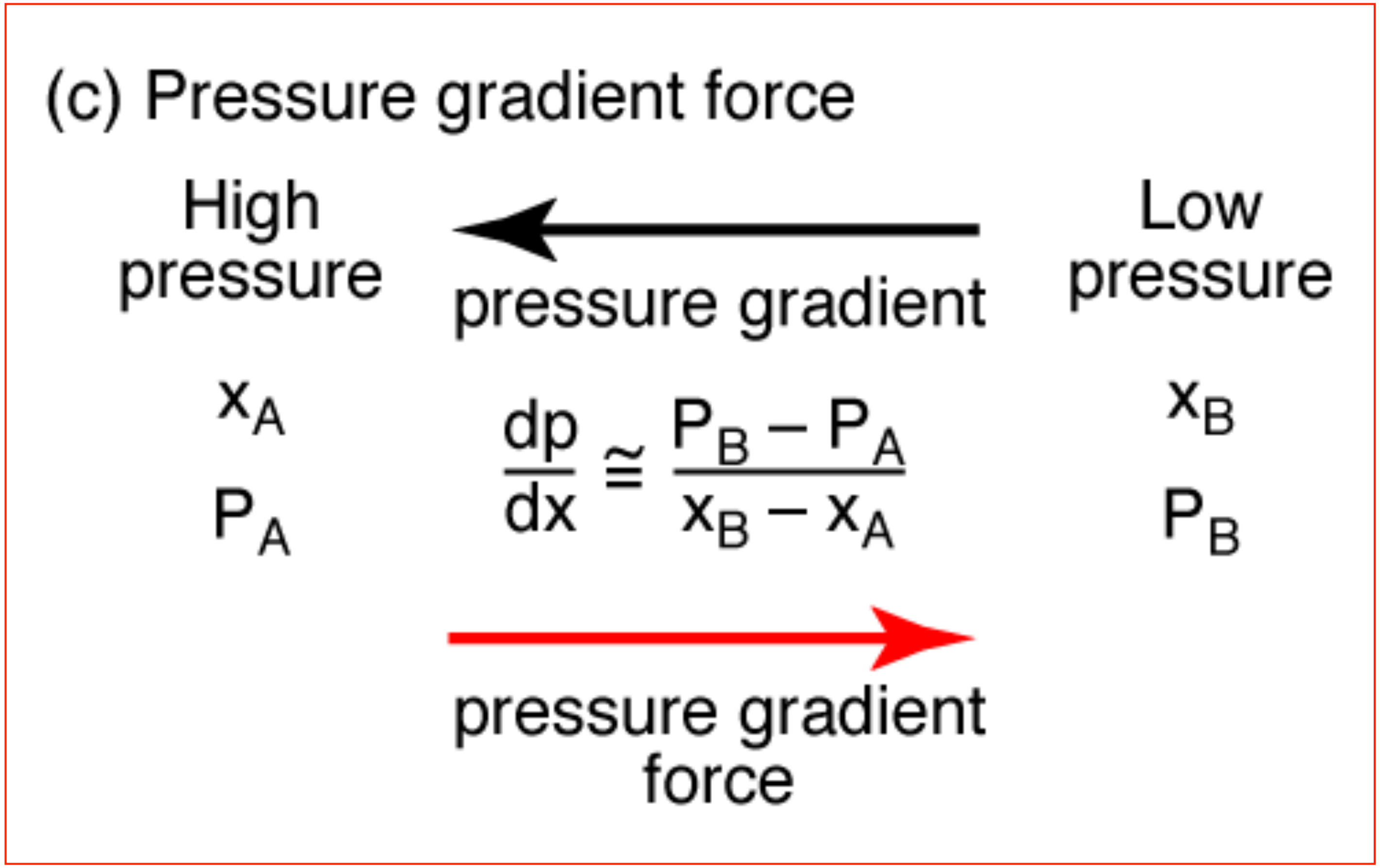
Geoid



http://www.esa.int/spaceinimages/Images/2011/03/New_GOCE_geoid

# Forces acting on geophysical fluid

1. Gravity
2. Pressure gradient force
3. Friction (dissipation) (viscous force)



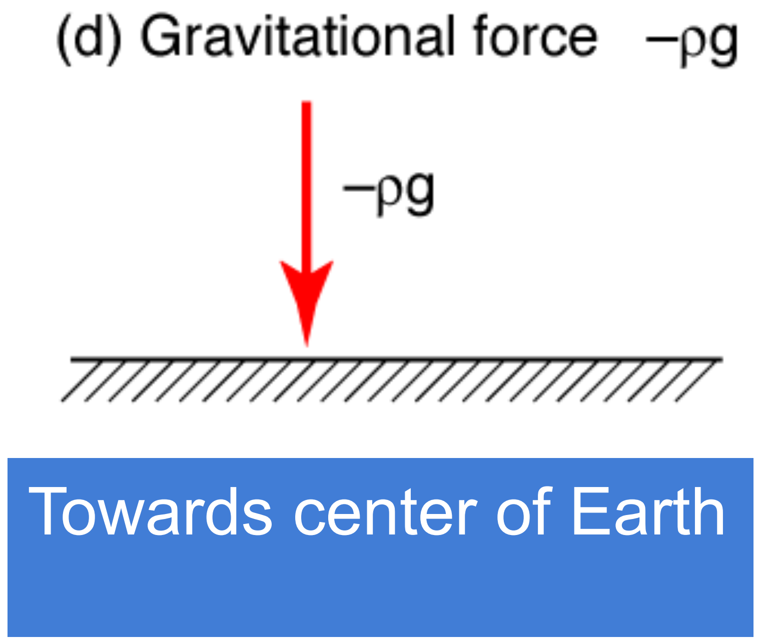

DPO Fig. 7.1

# Forces acting on geophysical fluid: vertical force balance

1. Gravity: g = 9.8 m/sec$^2$
2. Pressure gradient force
3. Friction (dissipation) (viscous force)

## Vertical balance (includes surface & internal waves):

Vertical acceleration +advection = pressure gradient force + gravity + viscous




**Hydrostatic balance:**

Dominant terms for many phenomena (not surface/ internal waves) – "static" – very small acceleration and advection
Viscous term is very small

0 = PGF + gravity (in words)

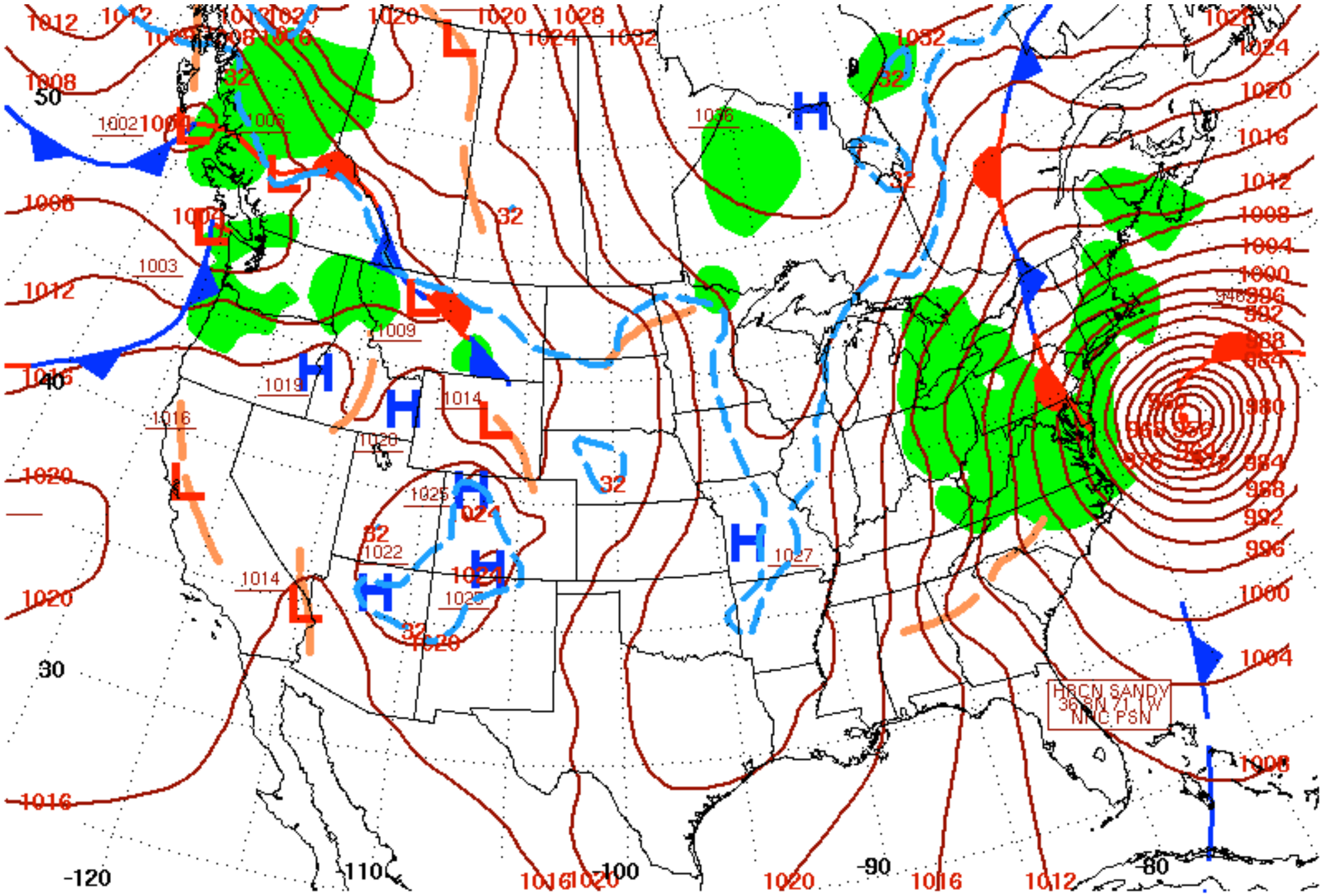
$0 = -\partial p/\partial z - \rho g$ (equation)

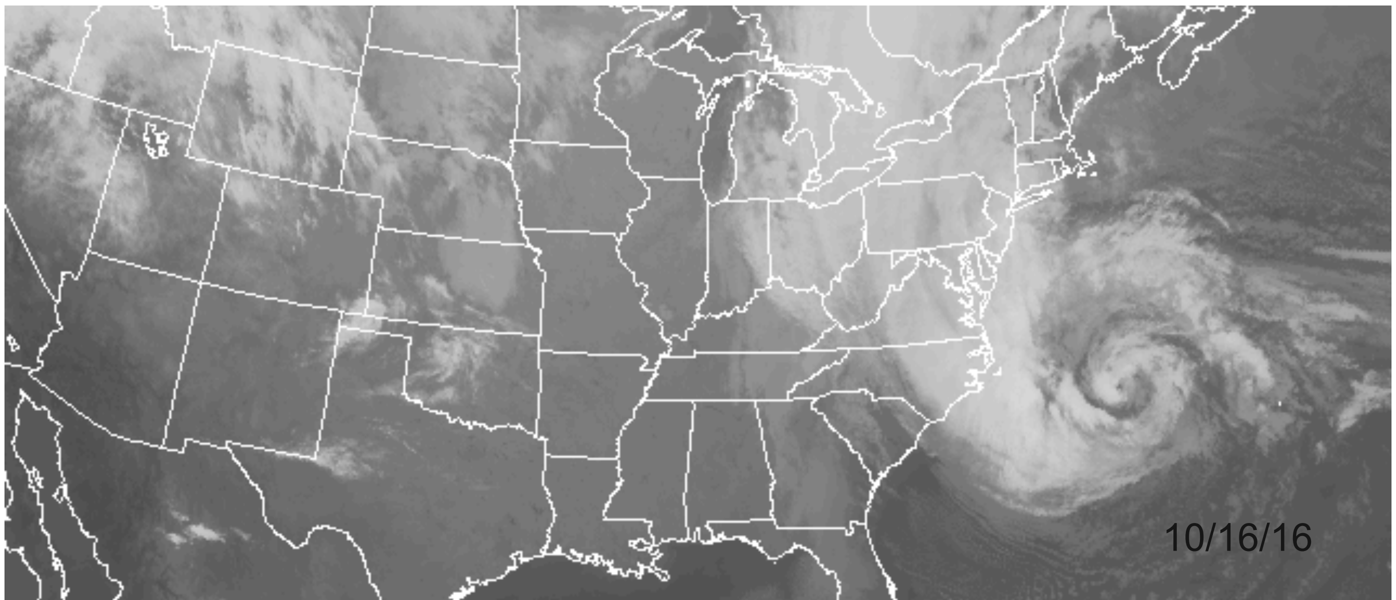
# Example of horizontal pressure gradients

## Daily surface pressure map

(and IR map showing clouds)

10/29/2012



(in the atmosphere, we can simply measure the pressure at the surface)

http://www.weather.gov

# Example of horizontal pressure gradients

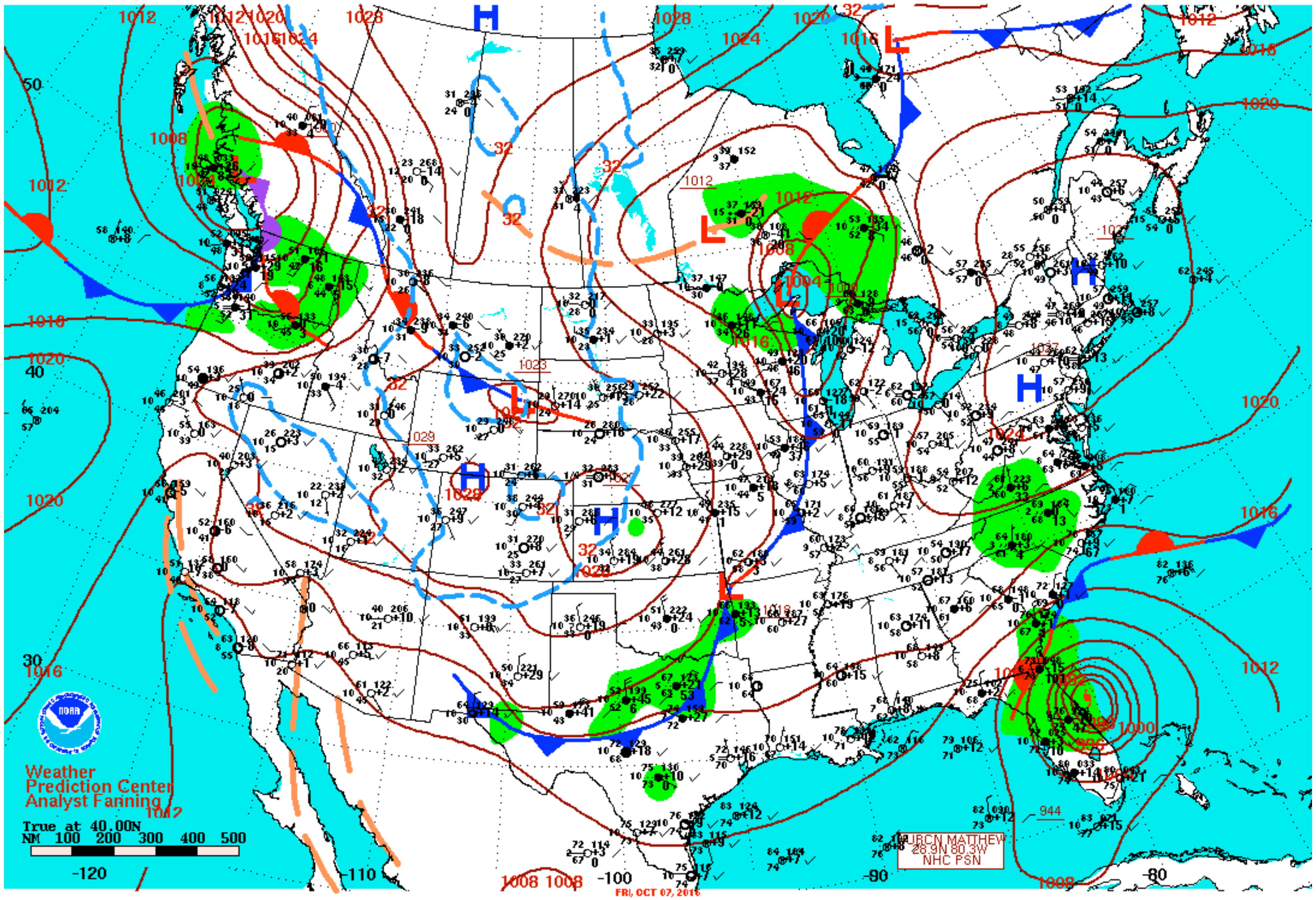
Daily surface pressure map

(and IR map showing clouds)

10/7/2016

(in the atmosphere, we can simply measure the pressure at the surface)

http://www.weather.gov

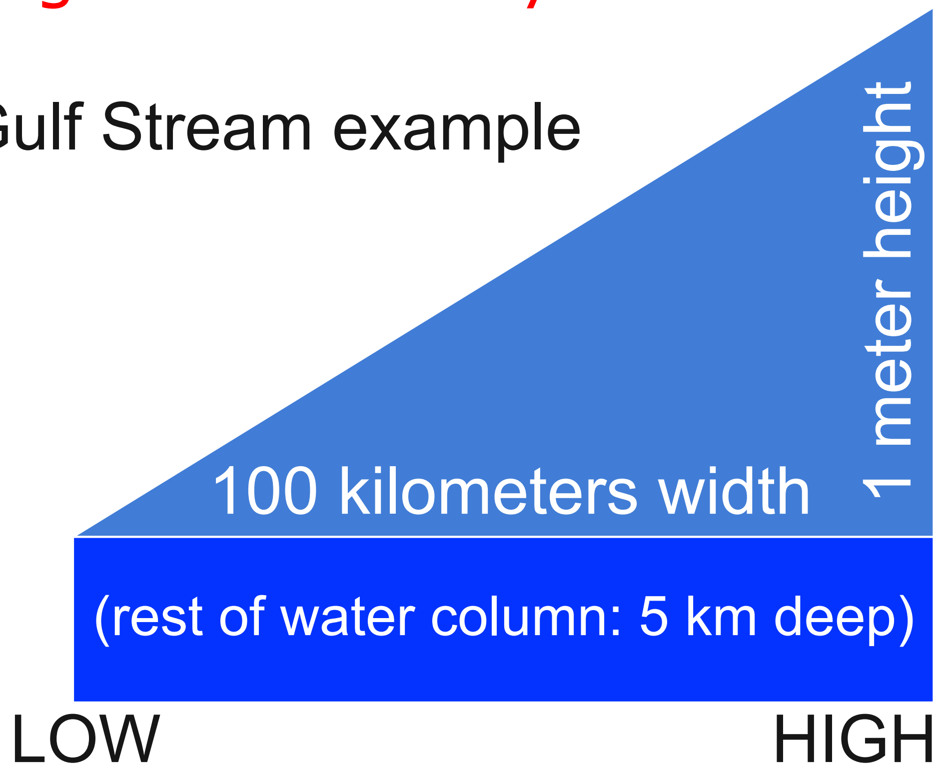



Surface Weather Map and Station Weather at 7:00 A.M. E.S.T.

# Horizontal pressure gradient force

- Small deviations of sea surface drive flows - usually much less than 1 meter height. Pressure gradient is very difficult to measure directly.

Gulf Stream example




1 meter height

Equivalent to pressure of 1 dbar, since water density is ~ 1000 kg/m$^3$

Pressure gradient is directed from low to high. Calculate size.

Pressure gradient force is directed from high to low (water pushed towards lower pressure).

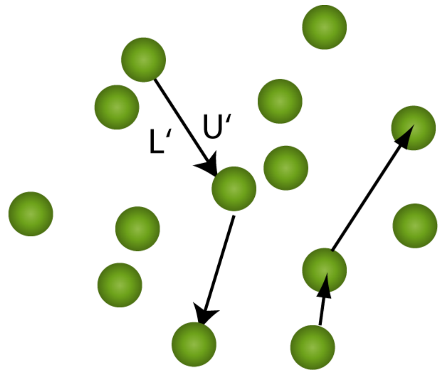
# Compute acceleration due to PGF

- Take the example of the Gulf Stream and compute the velocity after 1 year of acceleration.
- (To calculate this, note that the PGF is $-(1/\rho)\Delta p/\Delta x$ and that the pressure difference is given from the hydrostatic balance $-(1/\rho)\Delta p/\Delta z = g$; use $g=9.8$ m/s$^2$, $\Delta z=1$ m; $\Delta x = 100$ km. Then $\Delta u/\Delta t$ for acceleration, $\Delta t = 1$ year $\sim 3.14\text{x}10^7$ sec.)

- You'll find it's ridiculously large (compared with the observed 1 m/sec). How is such a large pressure gradient maintained without large velocities?
- (answer: Earth's rotation - Coriolis to be discussed next lecture)

# Mixing, diffusion, and viscosity

Random motion of molecules carries "stuff" around and redistributes it (mixing)




Fick's Law: net flux of "stuff" is proportional to its gradient

- Flux = $-\kappa(Q_a-Q_b)/(x_a-x_b)$ => $-\kappa\nabla Q$
- where $\kappa$ is the diffusivity

Units: [Flux] = [velocity][stuff], so

[$\kappa$] = [velocity][stuff][L]/[stuff] = [$L^2$/time] = $m^2$/sec

Assert (no derivation) for SIO 210 CSP: diffusive term has the form $\kappa\nabla^2 Q$

And the viscous force term has the form $\kappa\nabla^2 u$

# Values of molecular diffusivity and viscosity

- Molecular diffusivity and viscosity

  $\kappa_T$ = 0.0014 $cm^2$/sec (temperature)

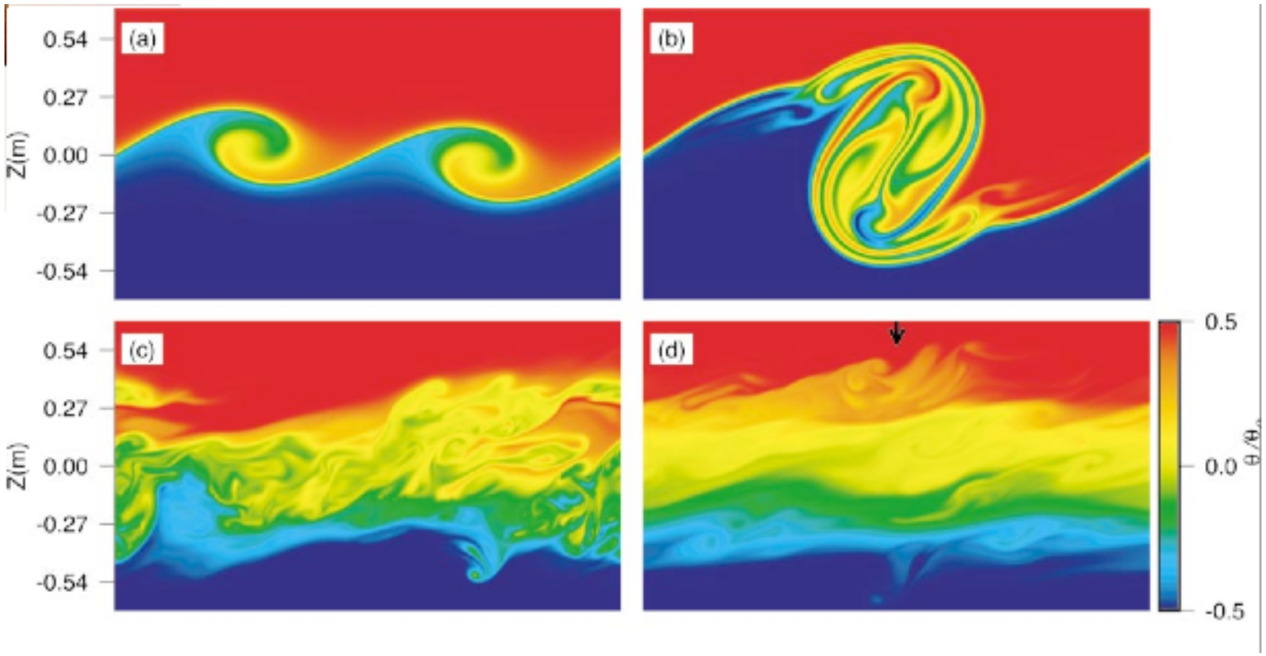
  $\kappa_S$ = 0.000013 $cm^2$/sec (salinity)

  $\nu$ = 0.018 $cm^2$/sec at 0°C (0.010 at 20°C)

These are very small values and have almost no effect.
How does the ocean (and atmosphere) actually mix? (next slides)

# Eddy diffusivity and eddy viscosity

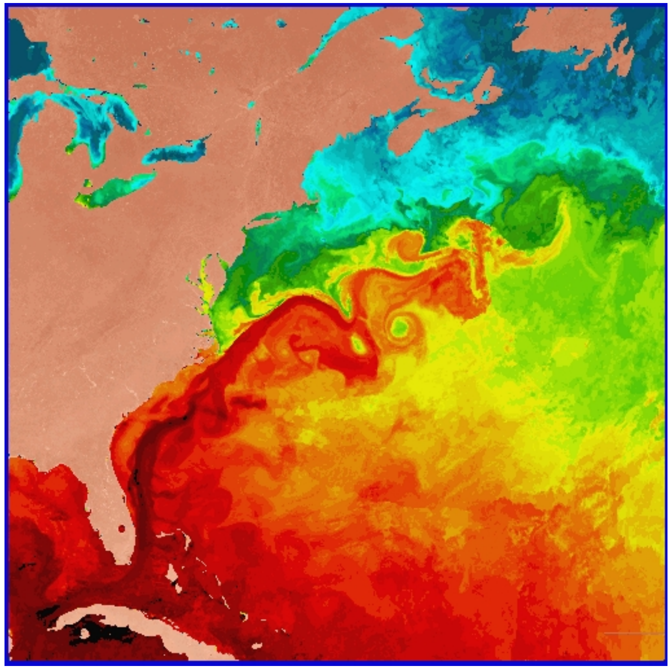
- Molecular viscosity and diffusivity are extremely small (values given on later slide)
- We know from observations that the ocean behaves as if diffusivity and viscosity are much larger than molecular (I.e. ocean is much more diffusive than this)
- The ocean has lots of turbulent motion (like any fluid)
- Turbulence acts on larger scales of motion like a viscosity - think of each random eddy or packet of waves acting like a randomly moving molecule carrying its property/mean velocity/information




# Stirring and mixing

Vertical stirring and ultimately mixing:

Internal waves on an interface stir fluid, break and mix

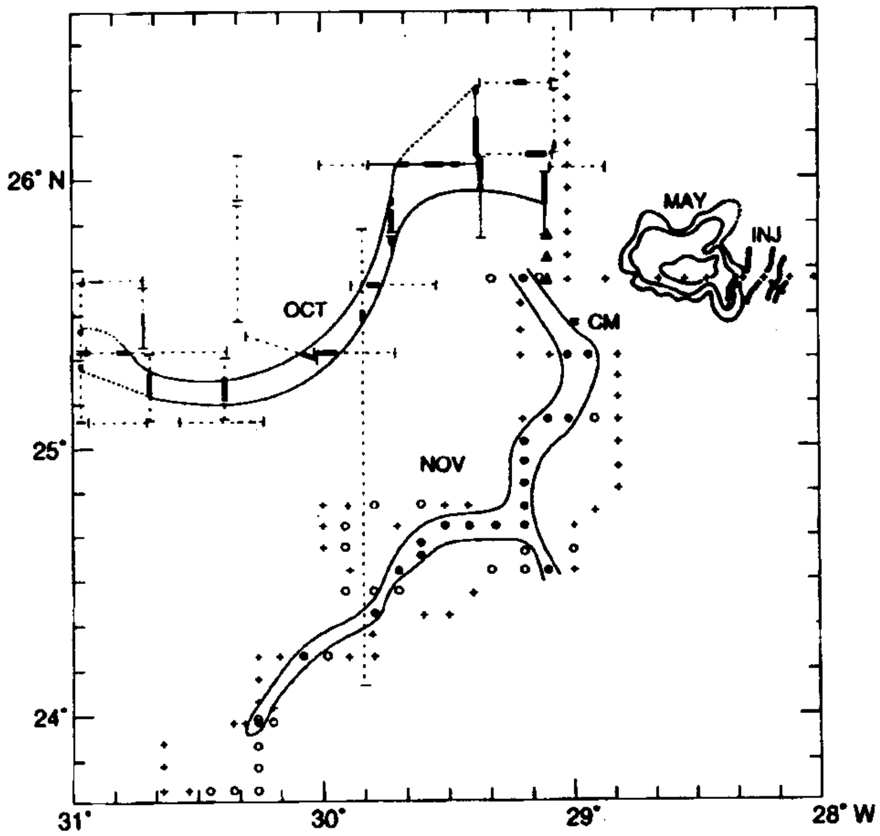


Horizontal stirring and ultimately mixing:

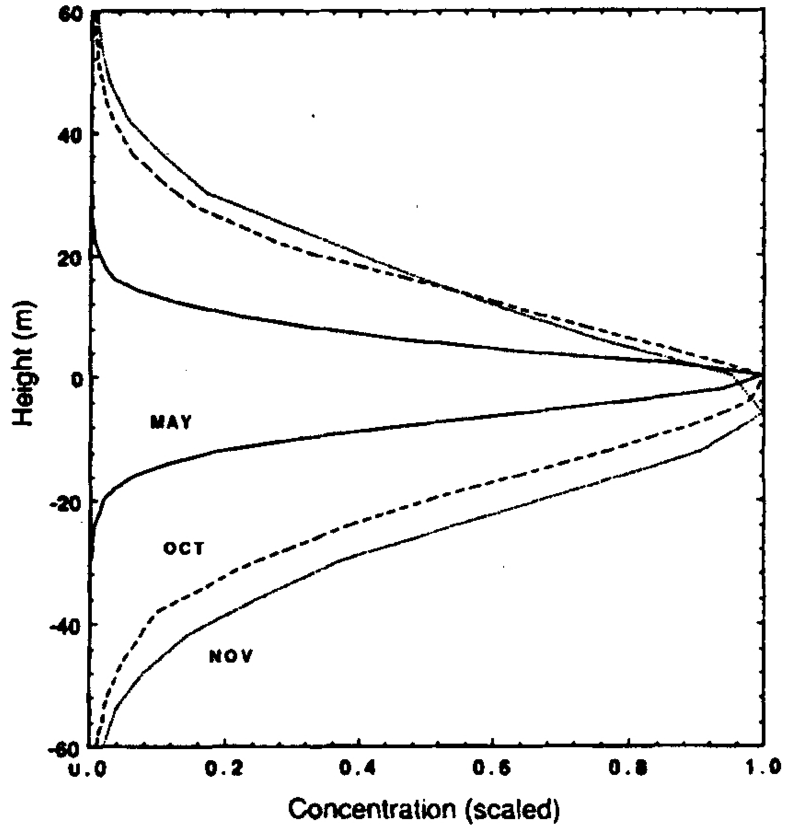
Gulf Stream (top): meanders and makes rings (closed eddies) that transport properties to a new location

# Measurements of mixing in ocean: horizontal and vertical diffusion are very different from each other and much larger than molecular diffusion

Horizontal diffusion




Vertical diffusion




- Intentional dye release, then track the dye over months

Ledwell et al Nature (1993)

# Values of molecular and eddy diffusivity and viscosity

- Molecular diffusivity and viscosity
  $\kappa_T = 0.0014 \text{ cm}^2/\text{sec}$ (temperature)
  $\kappa_S = 0.000013 \text{ cm}^2/\text{sec}$ (salinity)

  $\nu = 0.018 \text{ cm}^2/\text{sec}$ at 0°C (0.010 at 20°C)

- Eddy diffusivity and viscosity values for heat, salt, properties are the same size (same eddies carry momentum as carry heat and salt, etc)
  But eddy diffusivities and viscosities differ in the horizontal and vertical

- Eddy diffusivity and viscosity
  $A_H = 10^4$ to $10^8 \text{ cm}^2/\text{sec}$ (horizontal) = 1 to $10^4 \text{ m}^2/\text{sec}$
  $A_V = 0.1$ to $1 \text{ cm}^2/\text{sec}$ (vertical) = $10^{-5}$ to $10^{-4} \text{ m}^2/\text{sec}$

# Completed force balance (no rotation)

acceleration + advection =
pressure gradient force + gravity + viscous term

x: $\partial u/\partial t + u\, \partial u/\partial x + v\, \partial u/\partial y + w\, \partial u/\partial z =$
$- (1/\rho)\partial p/\partial x + \partial/\partial x(A_H \partial u/\partial x) +$
$\partial/\partial y(A_H \partial u/\partial y) + \partial/\partial z(A_V \partial u/\partial z)$

y: $\partial v/\partial t + u\, \partial v/\partial x + v\, \partial v/\partial y + w\, \partial v/\partial z =$
$- (1/\rho)\partial p/\partial y + \partial/\partial x(A_H \partial v/\partial x) +$
$\partial/\partial y(A_H \partial v/\partial y) + \partial/\partial z(A_V \partial v/\partial z)$

z: $\partial w/\partial t + u\, \partial w/\partial x + v\, \partial w/\partial y + w\, \partial w/\partial z =$
$- (1/\rho)\partial p/\partial z - g + \partial/\partial x(A_H \partial w/\partial x) +$
$\partial/\partial y(A_H \partial w/\partial y) + \partial/\partial z(A_V \partial w/\partial z)$

# Equations for temperature, salinity, density

- Temperature is changed by advection, heating, cooling, mixing (diffusion and double diffusion)
- Salinity is changed by advection, evaporation, precipitation/runoff, brine rejection during ice formation, mixing (diffusion and double diffusion)

- Density is related to temperature and salinity through the equation of state.
- Often we just write an equation for density change and ignore separate temperature, salinity

# Equations for temperature, salinity, density in words

T: change in T + advection of T =
heating source + diffusion of T

S: change in S + advection of S =
dilution by evaporation/precipitation + diffusion of S

$\rho$: $\rho = \rho(S,T,p)$
(relate density to T, S, p through equation of state)

# Equations for temperature, salinity, density in Δ format

T: $\Delta T/\Delta t + u(\Delta T/\Delta x) + v(\Delta T/\Delta y) + w(\Delta T/\Delta z) =$

$(Q/h)/(\rho c_p) + \Delta(\kappa_H \Delta T/\Delta x)/\Delta x$

$+ \Delta(\kappa_H \Delta T/\Delta y)/\Delta y + \Delta(\kappa_V \Delta T/\Delta z)/\Delta z$

S: $\Delta S/\Delta t + u(\Delta S/\Delta x) + v(\Delta S/\Delta y) + w(\Delta S/\Delta z) =$

$\mathcal{S} + \Delta(\kappa_H \Delta S/\Delta x)/\Delta x$

$+ \Delta(\kappa_H \Delta S/\Delta y)/\Delta y + \Delta(\kappa_V \Delta S/\Delta z)/\Delta z$

ρ: $\rho = \rho(S,T,p)$

Simplification: treat diffusivities $\kappa_H$ and $\kappa_V$ as constant, so diffusion terms become, e.g. $\kappa_H \Delta(\Delta T/\Delta x)/\Delta x$

Heating term: note that heat source is included as Q/h where is a thickness over which the heat is distibuted (units of Q are $W/m^2$)

Fine print: T and S diffusivities κ might not necessarily be equal ("double diffusion" in which development of stratification affected by differing diffusivities)

# Equations for temperature, salinity, density in differential equation format

T: $\partial T/\partial t + u\, \partial T/\partial x + v\, \partial T/\partial y + w\, \partial T/\partial z$

$$= (Q/h)/(\rho c_p) + \partial/\partial x(\kappa_H \partial T/\partial x) + \partial/\partial y(\kappa_H \partial T/\partial y) + \partial/\partial z(\kappa_V \partial T/\partial z)$$

$$\rightarrow (Q/h)/(\rho c_p) + \kappa_H(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2) + \kappa_V \partial^2 T/\partial z^2$$

S: $\partial S/\partial t + u\, \partial S/\partial x + v\, \partial S/\partial y + w\, \partial S/\partial z$

$$= S + \partial/\partial x(\kappa_H \partial S/\partial x) + \partial/\partial y(\kappa_H \partial S/\partial y) + \partial/\partial z(\kappa_V \partial S/\partial z)$$

$$\rightarrow S + \kappa_H(\partial^2 S/\partial x^2 + \partial^2 S/\partial y^2) + \kappa_V \partial^2 S/\partial z^2$$

ρ: $\rho = \rho(S,T,p)$

→Simplification: treat diffusivities $\kappa_H$ and $\kappa_V$ as constant

Heating term: note that heat source is included as Q/h where is a thickness over which the heat is distibuted (units of Q are $W/m^2$)

Fine print: T and S diffusivities κ might not necessarily be equal ("double diffusion" in which development of stratification affected by differing diffusivities)