SIO 210: Dynamics II: momentum balance (no rotation)

- Continuity (mass conservation) and Fick's Law
- Force balance
- Lecture emphasis: advection, pressure gradient force, eddy viscosity

Reading: DPO
Chapter 5.1

Chapter 7.1, 7.2
(skip 7.2.3)

Equations for fluid mechanics (for the ocean)

- Mass conservation (continuity) (no holes) (covered in previous lecture)

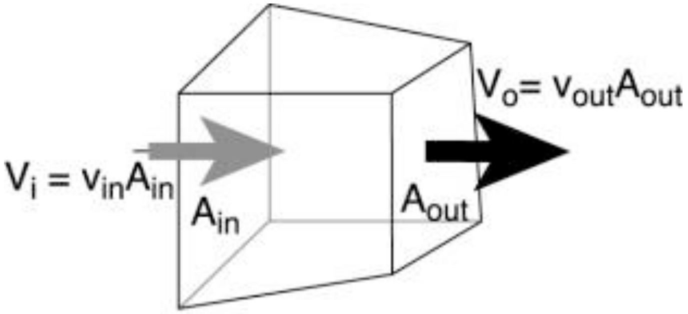
- Force balance: Newton's Law (**F**=m**a**) (3 equations)
- Equation of state (for oceanography, dependence of density on temperature, salinity and pressure) (1 equation)
- Equations for temperature and salinity change in terms of external forcing, or alternatively an equation for density change in terms of external forcing (2 equations). We have already looked at major aspects of these 2 in the transports lecture.

- 7 equations to govern it all

Review: Conservation of volume: Continuity at a point

Conservation at a point in the fluid (shrink the box to a point):

$$\nabla \bullet \vec{u} = 0$$



- 1D: $0 = \Delta u/\Delta x = \partial u/\partial x$
- 2D: $0 = \Delta u/\Delta x + \Delta v/\Delta y = \partial u/\partial x + \partial v/\partial y$
- 3D: $0 = \Delta u/\Delta x + \Delta v/\Delta y + \Delta w/\Delta z = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z$
- (Net convergence or divergence within the ocean results in mounding or lowering of sea surface, or within isopycnal layers, same thing) NO holes in the ocean

Force balance in a fluid

- Newton's law

  $\mathbf{F} = m\mathbf{a}$ (from physics class)

  This is a vector equation, with 3 equations for each of the three directions (x, y and z)

  $m\mathbf{a} = \mathbf{F}$ (for fluids)

- In a continuous fluid

  Divide by volume, so express in terms of density ρ and force per unit volume $\mathfrak{I}$:

  $\rho\mathbf{a} = \mathfrak{I}$

  Next "Acceleration" in a fluid has two terms: actual acceleration and advection

Time change and Acceleration

- Time change the change in stuff with time, for instance temperature T or heat Q = $\rho c_p T$:

$$\Delta T/\Delta t \Rightarrow \partial T/\partial t$$

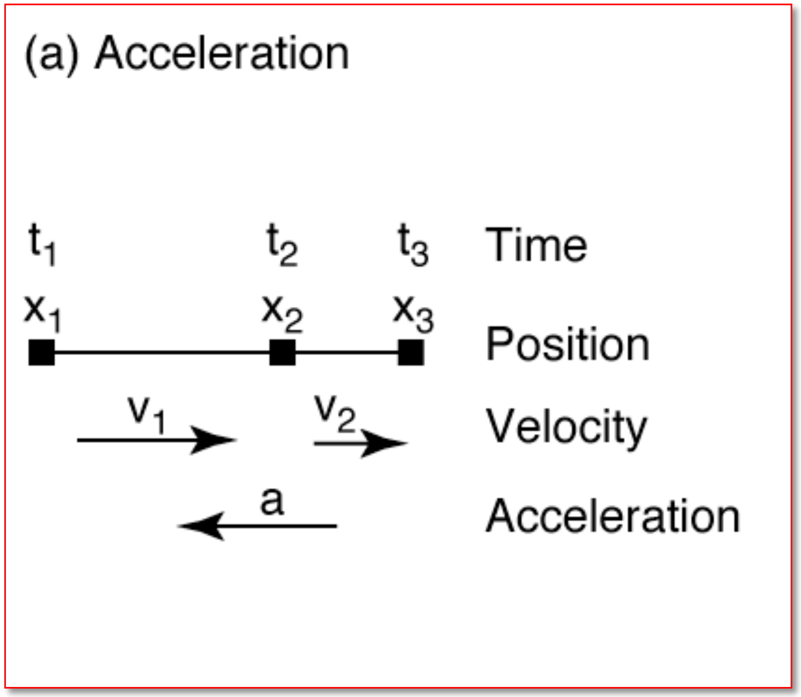
$$\Delta Q/\Delta t \Rightarrow \partial Q/\partial t$$

(Units are stuff/sec; here heat/sec or J/sec or W)

- Acceleration: the change in velocity with time

$$\Delta u/\Delta t \Rightarrow \partial u/\partial t$$

(Units are velocity/sec, hence m/sec^2)



DPO Fig. 7.1

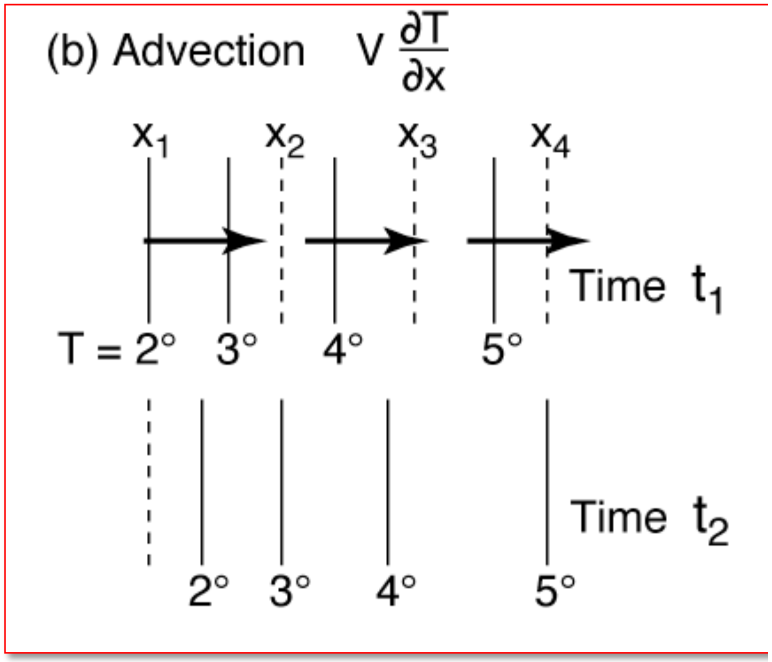
Advection

- Move "stuff" - temperature, salinity, oxygen, momentum, etc.
- By moving stuff, we might change the value of the stuff at the next location. We only change the value though if there is a difference ("gradient") in the stuff from one point to the next
- Advection is proportional to velocity and in the same direction as the velocity
- E.g. $u\,\Delta T/\Delta x$ or $u\,\partial T/\partial x$ is the advection of temperature in the x-direction
- Effect on time change of the property:

$$\partial T/\partial t = -u\,\partial T/\partial x$$

- Advection can act on velocity as well:

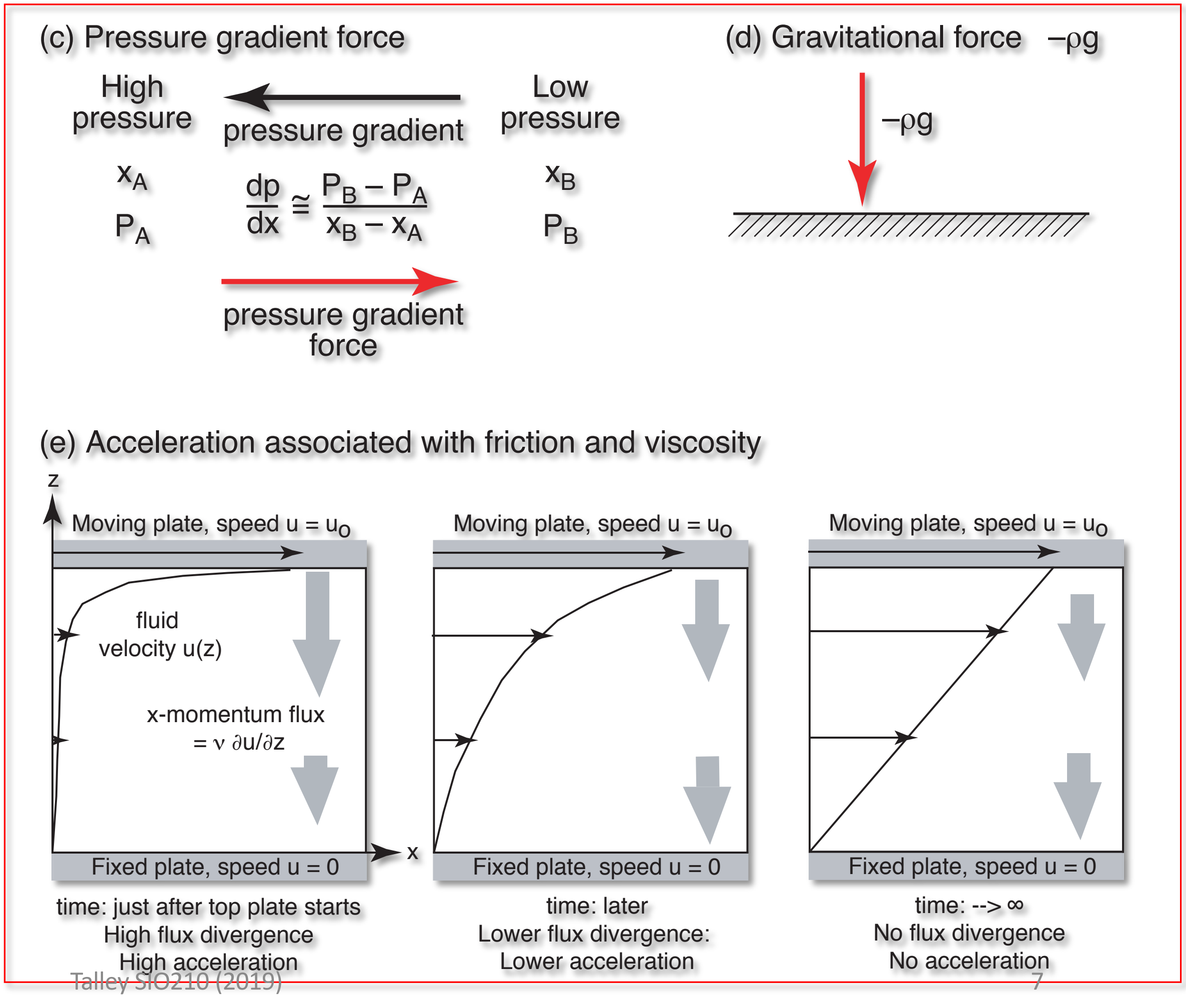
$$\partial u/\partial t = -u\,\partial u/\partial x$$



DPO Fig. 7.1

External forces acting on geophysical fluid

1. Gravity: $g = 9.8\ m/sec^2$
2. Pressure gradient force
3. Friction (dissipation)
   (viscous force)



DPO Fig. 7.1

Forces acting on geophysical fluid

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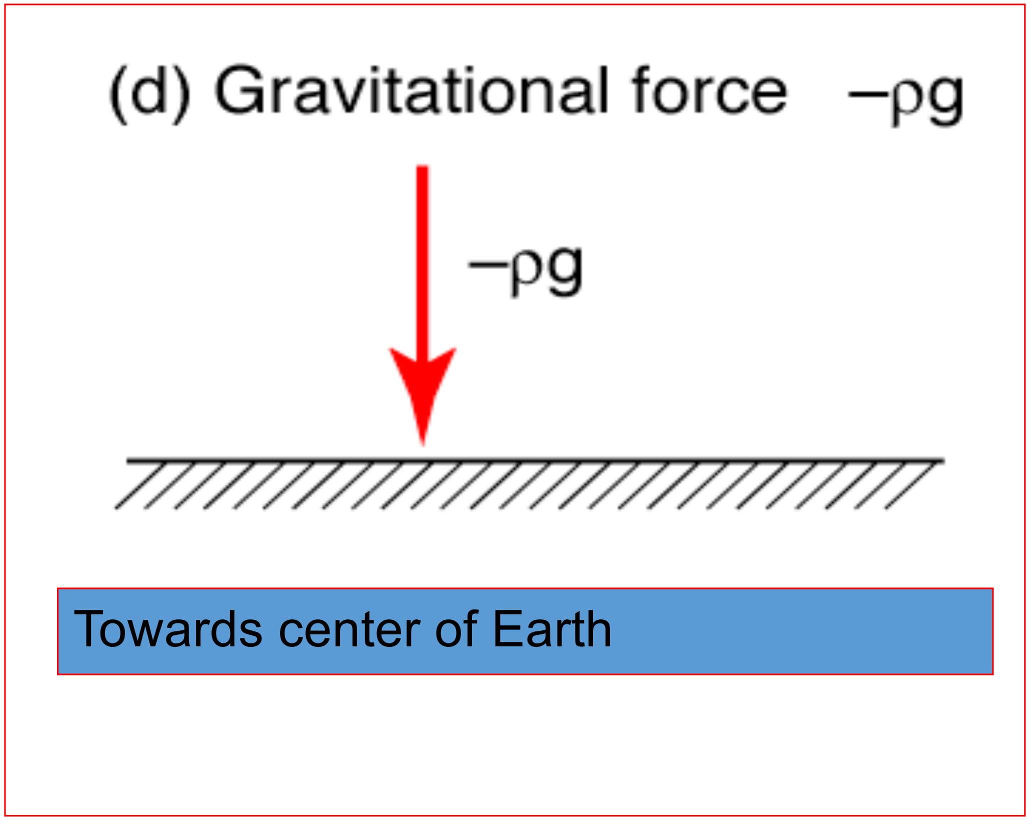
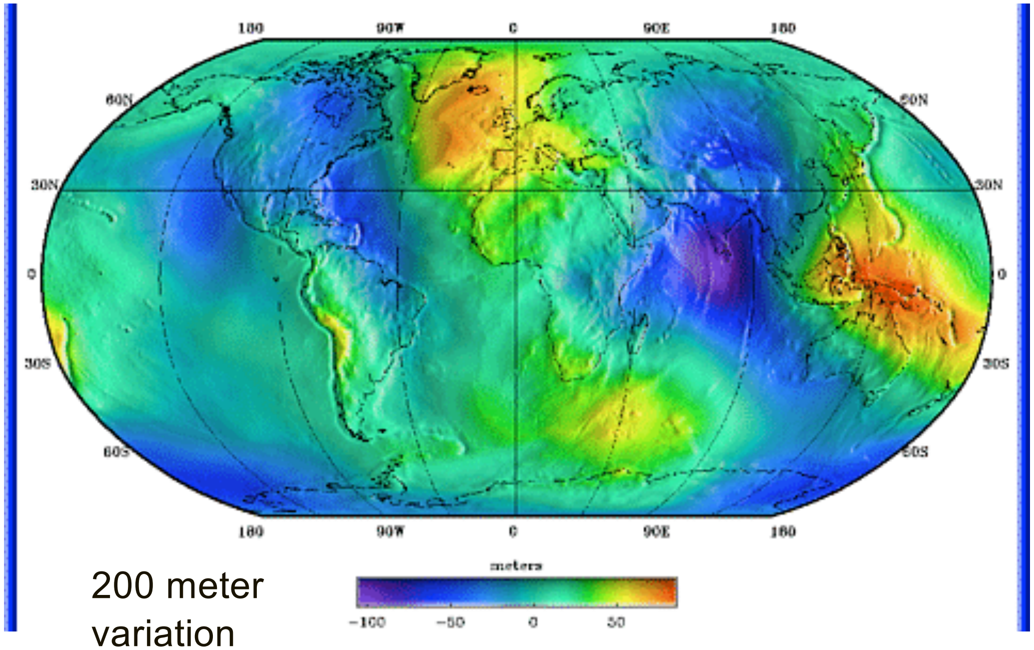


DPO Fig. 7.1

Gravity: Geoid

- Earth’s mass is not distributed evenly AND Earth is not a perfect ellipsoid
- http://www.ngs.noaa.gov/GEOID/



200 meter variation

Geoid map using **EGM96 d**ata, from http://cddis.gsfc.nasa.gov/926/egm96/egm96.html

Forces acting on geophysical fluid to here

1. Gravity: $g = 9.8\ \text{m/sec}^2$
2. Pressure gradient force
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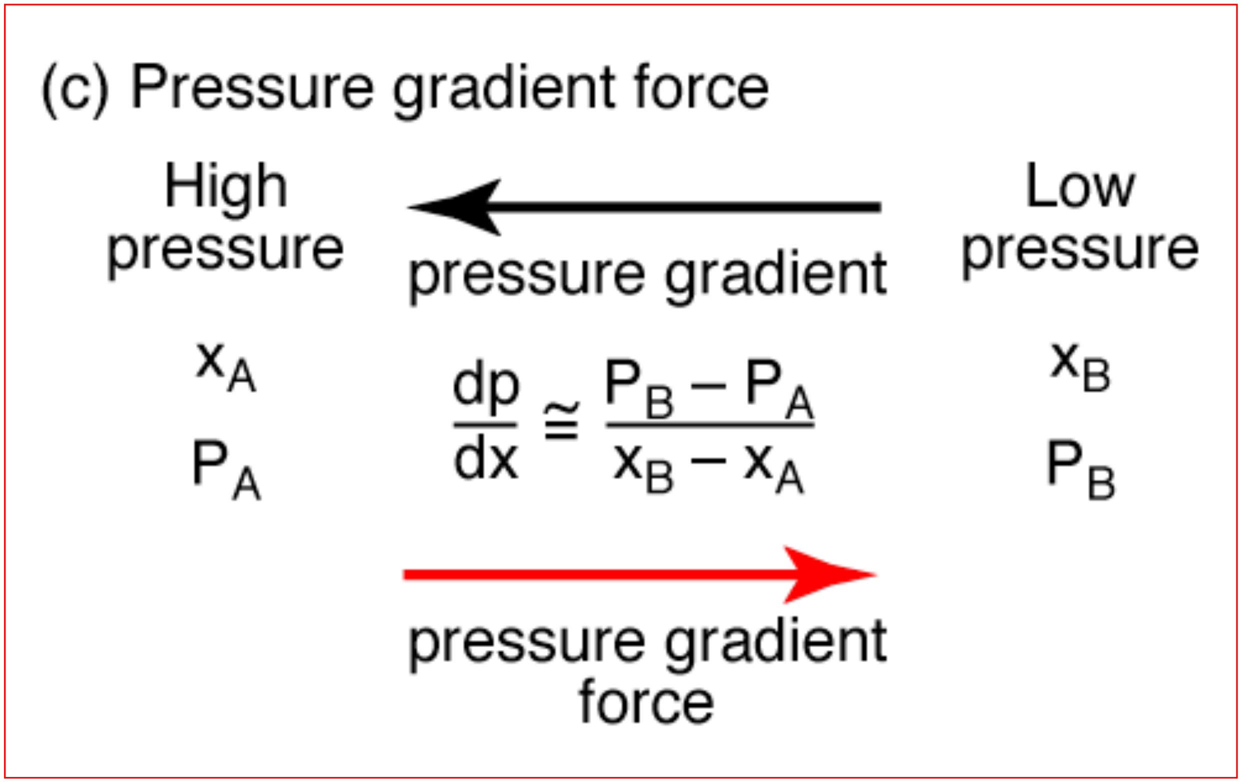
   (viscous force)



$$\frac{dp}{dx} \cong \frac{P_B - P_A}{x_B - x_A}$$

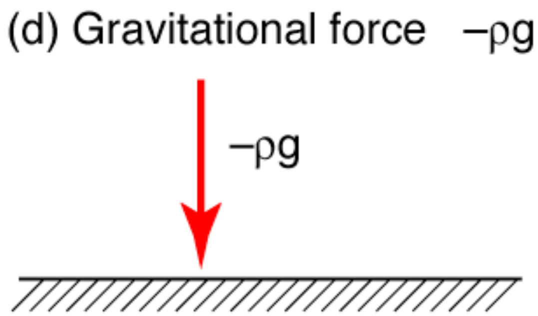
DPO Fig. 7.1

Vertical force balance (“Hydrostatic balance”)

1. Gravity: g = 9.8 m/sec^2
2. Pressure gradient force
3. Friction (dissipation) (viscous force)

Vertical balance:

Vertical acceleration +advection =
pressure gradient force + gravity + viscous



Towards center of Earth

Hydrostatic balance:

Dominant terms for many phenomena (not surface/internal waves) –
“static” – very small acceleration and advection
Viscous term is very small

0 = PGF + gravity (in words)

$0 = - \partial p/\partial z - \rho g$ (equation)

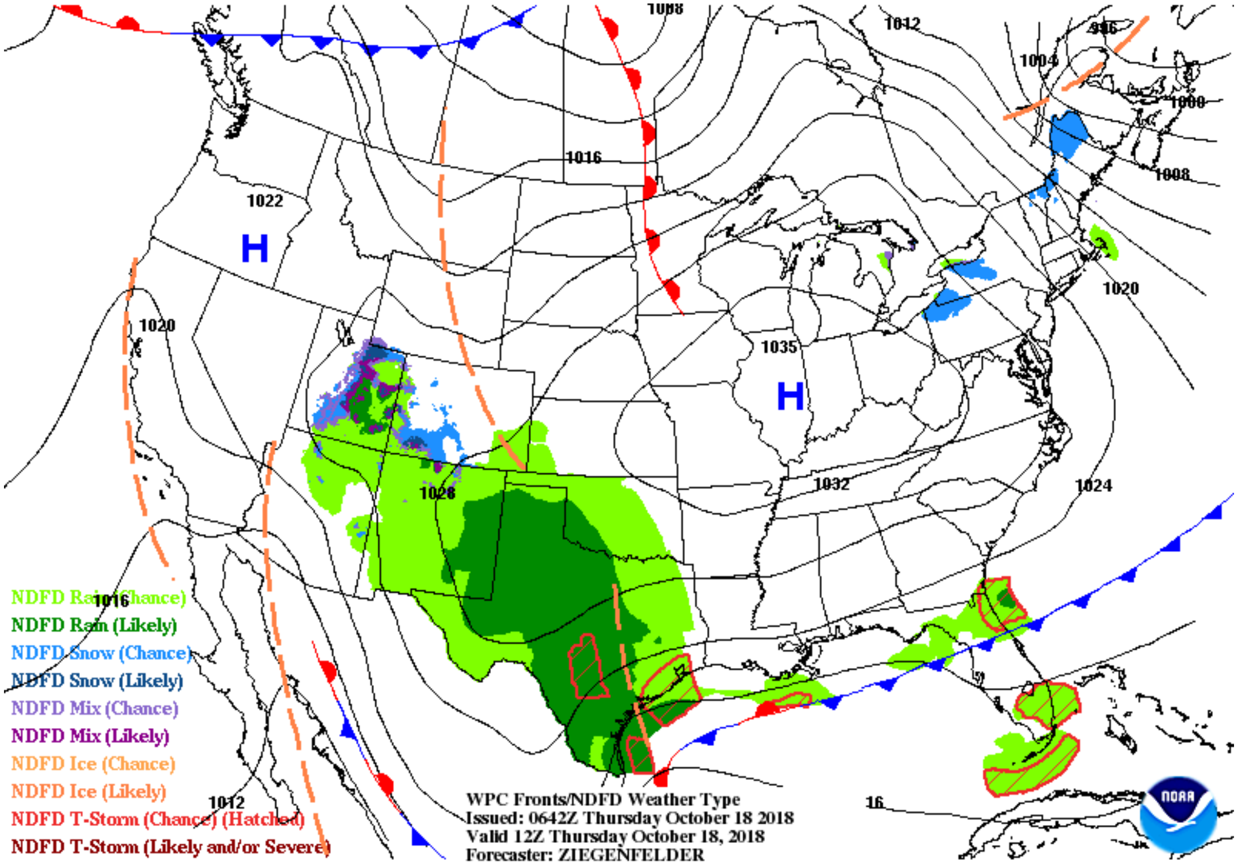
Horizontal forces

Horizontal force balance:

Horizontal acceleration +advection =
pressure gradient force + viscous

Atmosphere example:
Daily surface atmospheric pressure map: pressure gradients force the wind

(here Earth's rotation is important in understanding response to the pressure gradient force, ignore in this lecture)



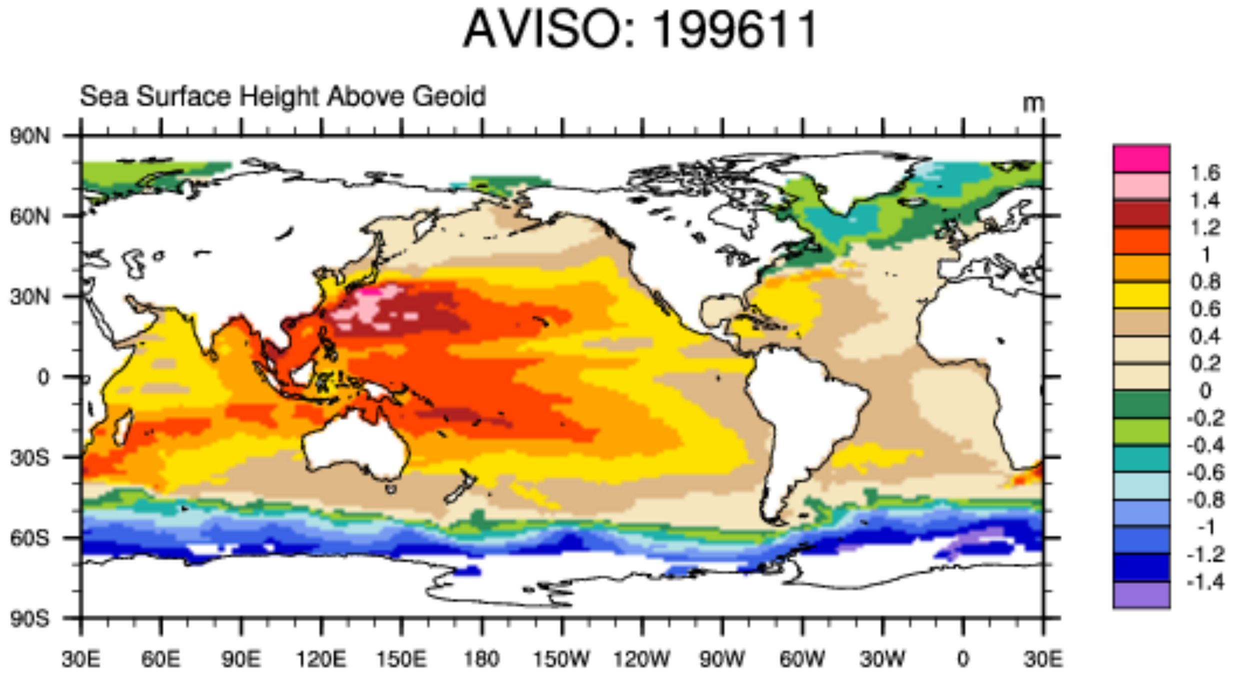
http://www.weather.gov

Horizontal forces

Horizontal force balance:
Horizontal acceleration +advection =
pressure gradient force + viscous

Ocean example
Ocean surface height map:
pressure gradients force the ocean currents

(here Earth's rotation is important in understanding response to the pressure gradient force, ignore in this lecture. Surface currents also involve viscous forces driven by wind stress.)

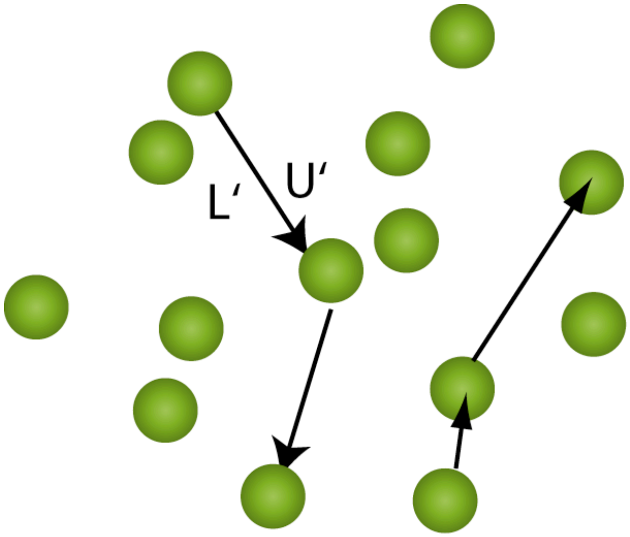


Sea surface height: satellite altimetry
https://climatedataguide.ucar.edu/climate-data/aviso-satellite-derived-sea-surface-height-above-geoid

Mixing, diffusion, and viscosity

- Random motion of molecules carries “stuff” around and redistributes it (mixes)

- Fick’s Law: net flux of “stuff” is proportional to its gradient
  - Flux = $-\kappa(Q_a-Q_b)/(x_a-x_b) \Rightarrow -\kappa\nabla Q$
  - where κ is the diffusivity



Units: [Flux] = [velocity][stuff], so

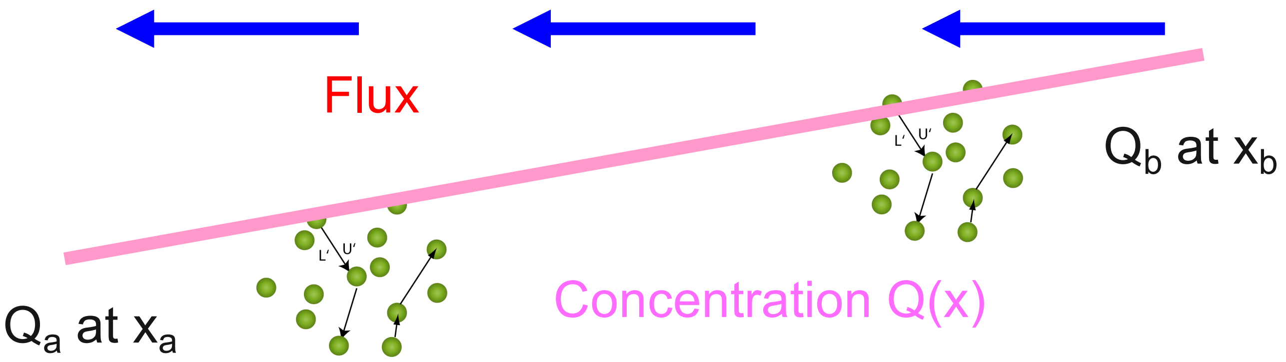
$[\kappa]$ = [velocity][stuff][L]/[stuff] =

$[L^2/\text{time}] = m^2/\text{sec}$

Mixing, diffusion, and viscosity

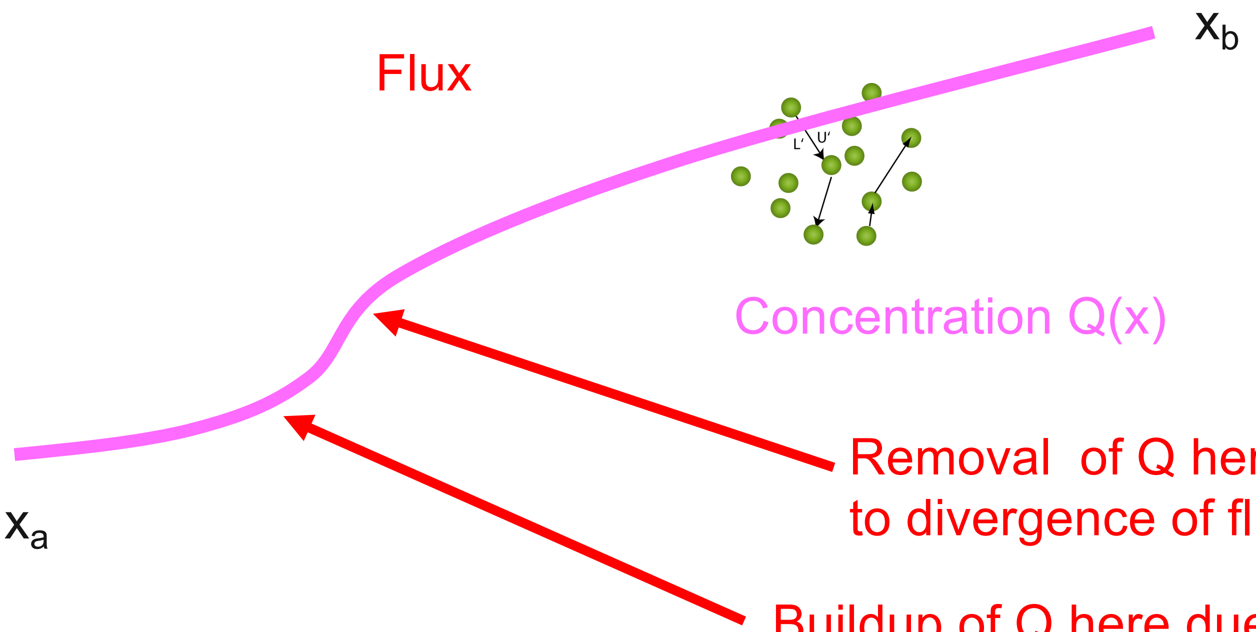
Fick's Law: flux of "stuff" is proportional to its gradient $\text{Flux} = -\kappa\nabla Q$

- If the concentration is exactly linear, with the amount of stuff at both ends maintained at an exact amount, then the flux of stuff is the same at every point between the ends, and there is no change in concentration of stuff at any point in between.



Mixing, diffusion, and viscosity

Diffusion: if there is a convergence or divergence of flux then the "stuff" concentration can change



Flux

Concentration Q(x)

Removal of Q here due to divergence of flux

Buildup of Q here due to convergence of flux

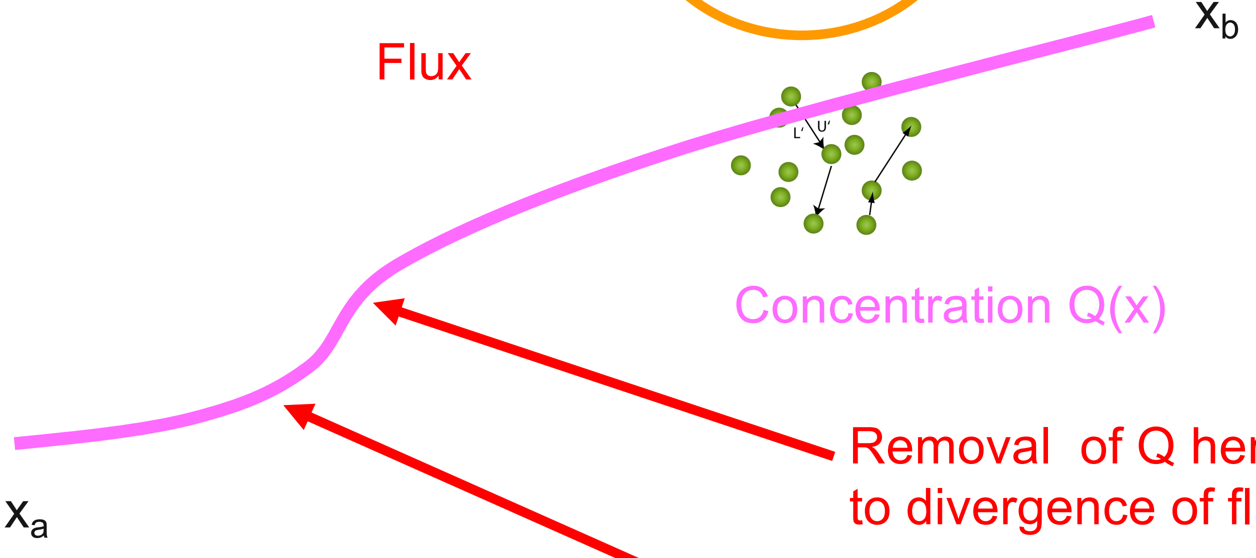
Mixing, diffusion, and viscosity

Change in Q with time = change in Q flux with space

$$\Delta Q/\Delta t = -\Delta \mathrm{Flux}/\Delta x$$

$$\partial Q/\partial t = \kappa\, \partial^2 Q/\partial x^2$$

Diffusion term



Flux

Concentration Q(x)

Removal of Q here due to divergence of flux

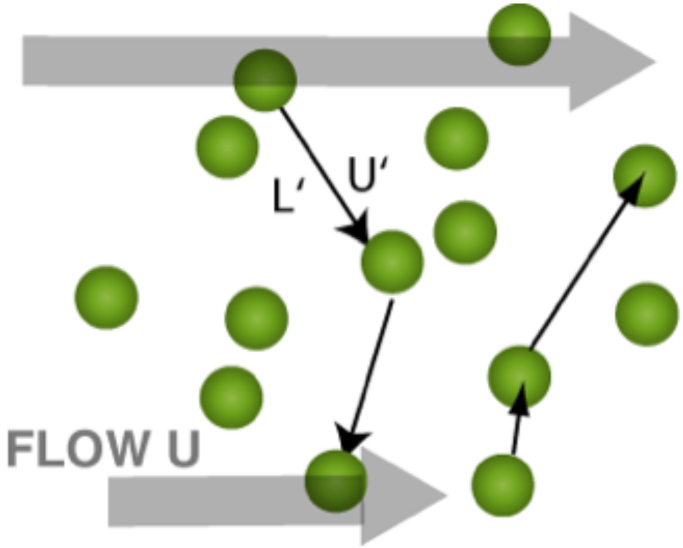
Buildup of Q here due to convergence of flux

Viscosity

- Viscosity: apply same Fick's Law concept to velocity. So viscosity affects flow if there is a convergence of flux of momentum.
- Stress ("flux of momentum") on flow is

$$\tau \text{ (= "flux") } = -\rho\upsilon\nabla u$$

where υ is the viscosity coefficient in m^2/sec



DPO Fig. 7.3

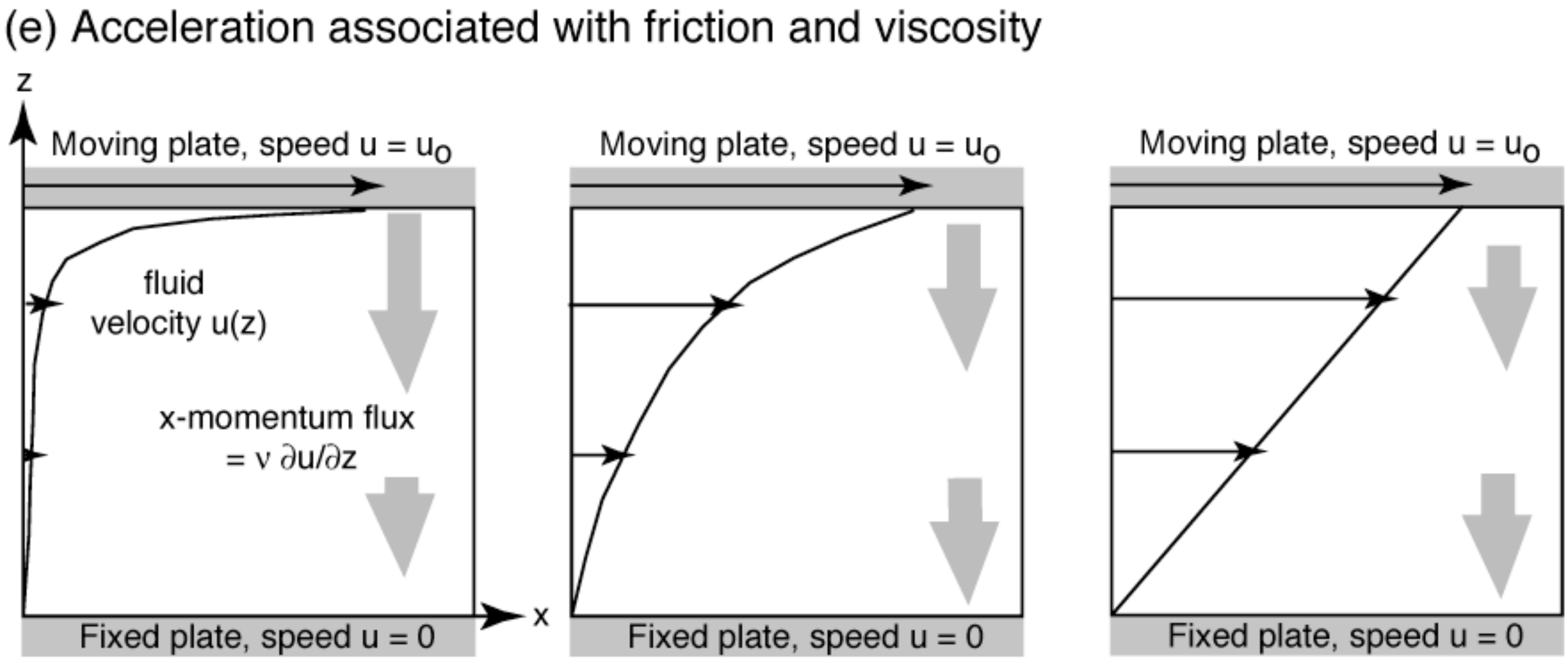
Aside:

υ is the kinematic viscosity

$\mu = \rho\upsilon$ is the dynamic viscosity in kg/m sec

Stress proportional to strain (shear) -> Newtonian fluid

Acceleration due to viscosity



time: just after top plate starts
High flux divergence
High acceleration

time: later
Lower flux divergence:
Lower acceleration

time: --> ∞
No flux divergence
No acceleration (no force)

DPO Fig. 7.1

Acceleration due to viscosity

- If the viscous momentum flux is convergent (or divergent) then it can accelerate the flow
- That is, if $\partial\tau/\partial x = -\partial(\rho\upsilon\ \partial u/\partial x) \neq 0$, then
- $\partial u/\partial t = \ \rho\upsilon\ \partial^2 u/\partial x^2 \underline{=}\ \mu\ \partial^2 u/\partial x^2$ if μ is constant

If the viscosity itself depends on space, then it needs to be INSIDE the space derivative: $\partial_x (\mu \partial u/\partial x)$

Values of molecular diffusivity and viscosity

- Molecular diffusivity and viscosity

  $\kappa_T = 0.0014\ \mathrm{cm^2/sec}$ (temperature)
  $\kappa_S = 0.000013\ \mathrm{cm^2/sec}$ (salinity)

  $\nu = 0.018\ \mathrm{cm^2/sec}$ at 0°C (0.010 at 20°C)

These are very small values and have almost no effect.
How does the ocean (and atmosphere) actually mix? (next slides)

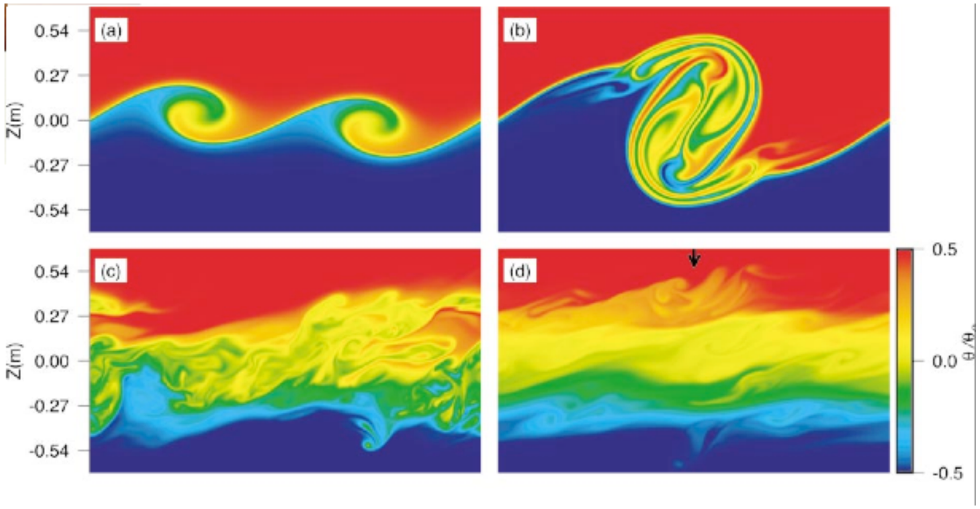
Eddy diffusivity and eddy viscosity

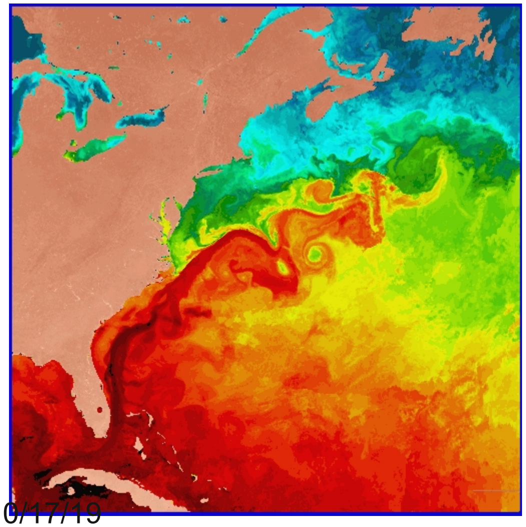
- Molecular viscosity and diffusivity are extremely small (values given on later slide)
- We know from observations that the ocean behaves as if diffusivity and viscosity are much larger than molecular (I.e. ocean is much more diffusive than this)
- The ocean has lots of turbulent motion (like any fluid)
- Turbulence acts on larger scales of motion like a viscosity - think of each random eddy or packet of waves acting like a randomly moving molecule carrying its property/mean velocity/information

Stirring and mixing

Vertical stirring and ultimately mixing:

Internal waves on an interface stir fluid, break and mix



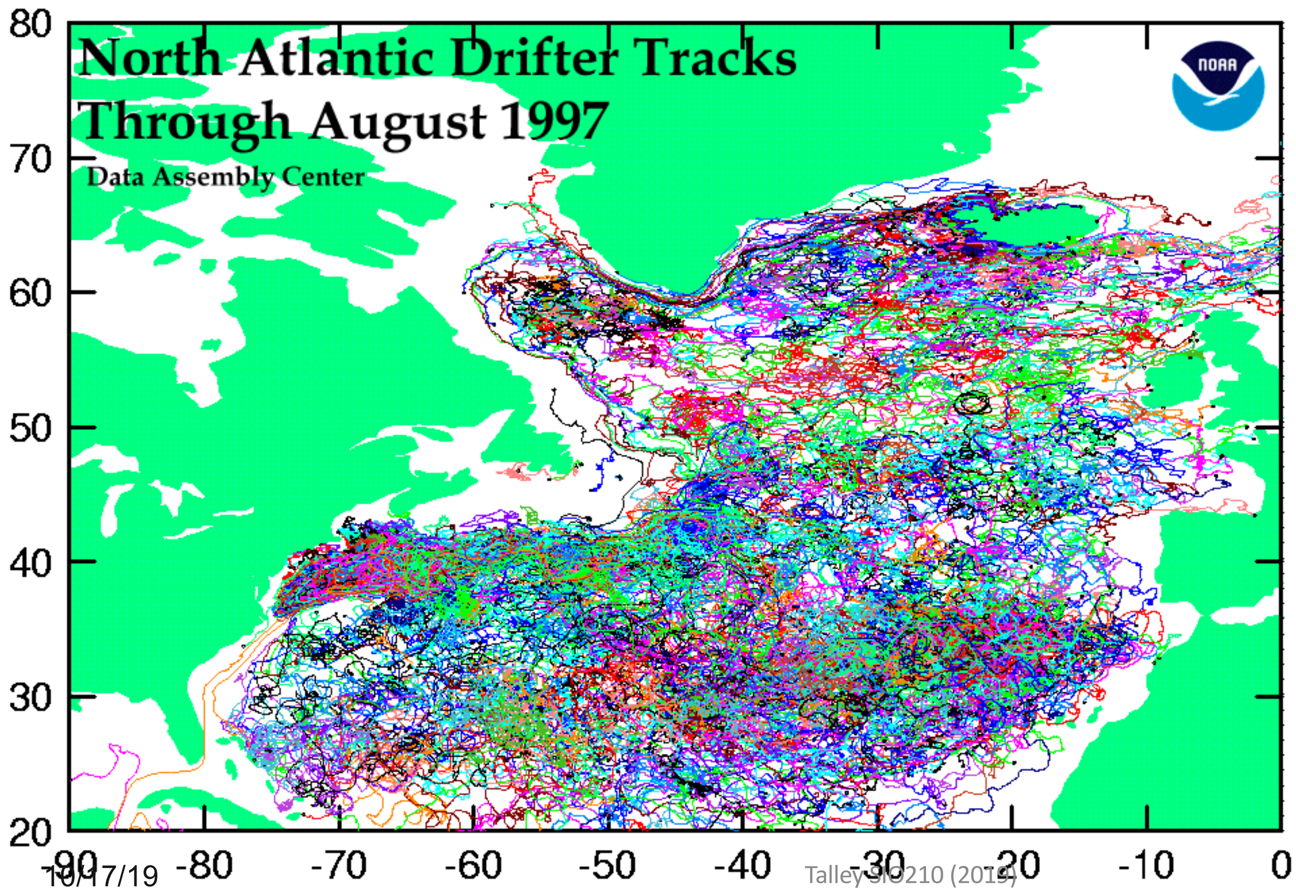


Horizontal stirring and ultimately mixing:

Gulf Stream: meanders and makes rings (closed eddies) that transport properties to a new location

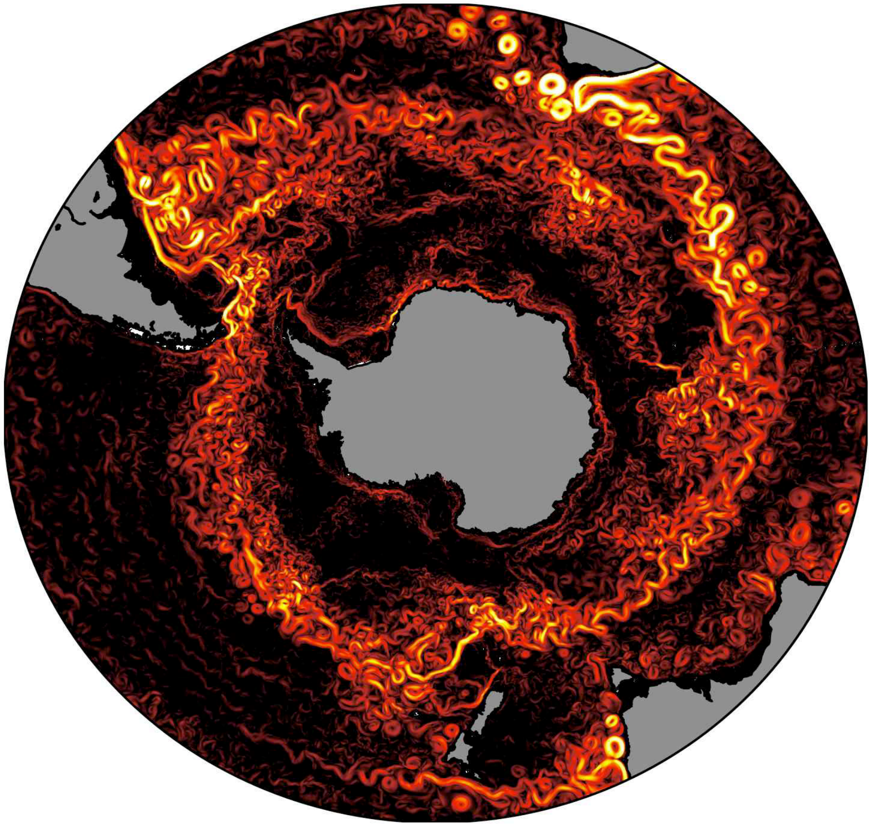
Eddy diffusivity and viscosity

Example of surface drifter tracks: dominated to the eye by variability
(they can be averaged to make a very useful mean circulation)



Contributing to lateral (horizontal) eddy diffusivity

Eddy field in a numerical model of the ocean

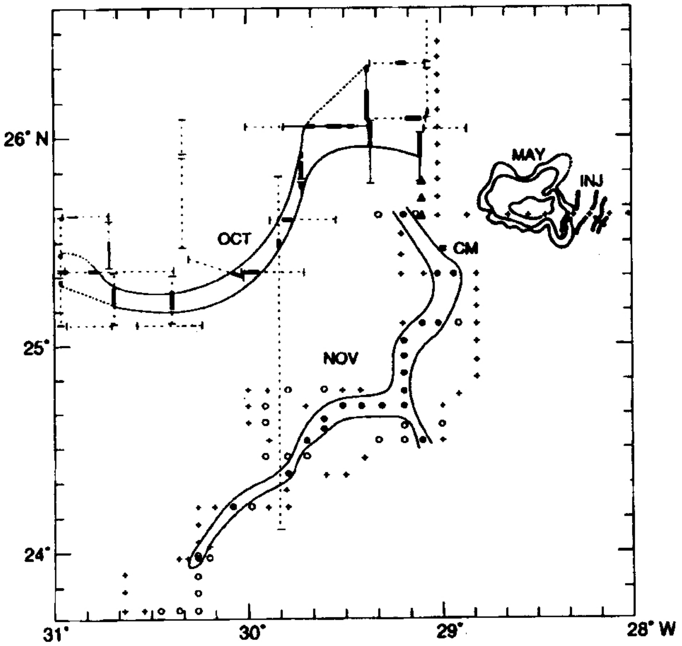


Contributing to lateral (horizontal) eddy diffusivity

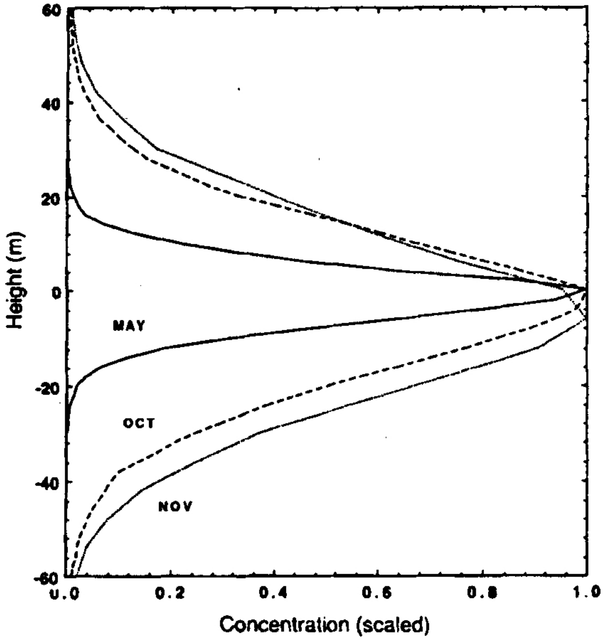
A.Morrison, GFDL CM2.6 model surface velocity

Measurements of mixing in ocean: horizontal and vertical diffusion are very different from each other and much larger than molecular diffusion

Horizontal diffusion



Vertical diffusion



- Intentional dye release, then track the dye over months

Ledwell et al Nature (1993)

Values of molecular and eddy diffusivity and viscosity

- Molecular diffusivity and viscosity

  κ_T = 0.0014 cm^2/sec   (temperature)
  κ_S = 0.000013 cm^2/sec (salinity)

  ν = 0.018 cm^2/sec at 0°C (0.010 at 20°C)

- Eddy diffusivity and viscosity values for heat, salt, properties are the same size (same eddies carry momentum as carry heat and salt, etc)

  But eddy diffusivities and viscosities differ in the horizontal and vertical

- Eddy diffusivity and viscosity

  A_H = 10^4 to 10^8 cm^2/sec (horizontal) = 1 to 10^4 m^2/sec
  A_V = 0.1 to 1 cm^2/sec (vertical) = 10^{-5} to 10^{-4} m^2/sec

Completed force balance (no rotation)

acceleration + advection = pressure gradient force + viscous term + gravity

x: $\partial u/\partial t + u\, \partial u/\partial x + v\, \partial u/\partial y + w\, \partial u/\partial z = -\, (1/\rho)\partial p/\partial x + \partial/\partial x(A_H \partial u/\partial x) +$
$\partial/\partial y(A_H \partial u/\partial y) + \partial/\partial z(A_V \partial u/\partial z)$

y: $\partial v/\partial t + u\, \partial v/\partial x + v\, \partial v/\partial y + w\, \partial v/\partial z = -\, (1/\rho)\partial p/\partial y + \partial/\partial x(A_H \partial v/\partial x) +$
$\partial/\partial y(A_H \partial v/\partial y) + \partial/\partial z(A_V \partial v/\partial z)$

z: $\partial w/\partial t + u\, \partial w/\partial x + v\, \partial w/\partial y + w\, \partial w/\partial z = -\, (1/\rho)\partial p/\partial z - g + \partial/\partial x(A_H \partial w/\partial x) +$
$\partial/\partial y(A_H \partial w/\partial y) + \partial/\partial z(A_V \partial w/\partial z)$

Equations for temperature, salinity, density

- Temperature is changed by advection, heating, cooling, mixing (diffusion and double diffusion)
- Salinity is changed by advection, evaporation, precipitation/runoff, brine rejection during ice formation, mixing (diffusion and double diffusion)

- Density is related to temperature and salinity through the equation of state.
- Often we just write an equation for density change and ignore separate temperature, salinity

Equations for temperature, salinity, density in words

T: change in T + advection of T =
heating source + diffusion of T

S: change in S + advection of S =
dilution by evaporation/precipitation + diffusion of S

ρ: $\rho = \rho(S,T,p)$ (relate density to T, S, p through equation of state)

Fine print: T and S diffusivities κ might not necessarily be equal ("double diffusion" in which development of stratification affected by differing diffusivities)

Equations for temperature, salinity, density in Δ format

$$\text{T:}\quad \Delta T/\Delta t + u(\Delta T/\Delta x) + v(\Delta T/\Delta y) + w(\Delta T/\Delta z) = (Q/h)/(\rho c_p) + \Delta(\kappa_H \Delta T/\Delta x)/\Delta x + \Delta(\kappa_H \Delta T/\Delta y)/\Delta y + \Delta(\kappa_V \Delta T/\Delta z)/\Delta z$$

$$\text{S:}\quad \Delta S/\Delta t + u(\Delta S/\Delta x) + v(\Delta S/\Delta y) + w(\Delta S/\Delta z) = S + \Delta(\kappa_H \Delta S/\Delta x)/\Delta x + \Delta(\kappa_H \Delta S/\Delta y)/\Delta y + \Delta(\kappa_V \Delta S/\Delta z)/\Delta z$$

$$\rho\text{:}\quad \rho = \rho(S,T,p)$$

Simplification: treat diffusivities κ_H and κ_V as constant, so diffusion terms become, e.g. $\kappa_H\Delta(\Delta T/\Delta x)/\Delta x$

Heating term: note that heat source is included as Q/h where h is a thickness over which the heat is distributed (units of Q are W/m^2)

Fine print: T and S diffusivities κ might not necessarily be equal ("double diffusion" in which development of stratification affected by differing diffusivities)

Equations for temperature, salinity, density in differential form

T: $\partial T/\partial t + u\,\partial T/\partial x + v\,\partial T/\partial y + w\,\partial T/\partial z = (Q/h)/(\rho c_p) + \partial/\partial x(\kappa_H \partial T/\partial x) +$

$\partial/\partial y(\kappa_H \partial T/\partial y) + \partial/\partial z(\kappa_V \partial T/\partial z)$

$\rightarrow (Q/h)/(\rho c_p) + \kappa_H(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2) + \kappa_V \partial^2 T/\partial z^2$

S: $\partial S/\partial t + u\,\partial S/\partial x + v\,\partial S/\partial y + w\,\partial S/\partial z = S + \partial/\partial x(\kappa_H \partial S/\partial x) +$

$\partial/\partial y(\kappa_H \partial S/\partial y) + \partial/\partial z(\kappa_V \partial S/\partial z)$

$\rightarrow S + \kappa_H(\partial^2 S/\partial x^2 + \partial^2 S/\partial y^2) + \kappa_V \partial^2 S/\partial z^2$

ρ: $\rho = \rho(S,T,p)$

→Simplification: treat diffusivities κ_H and κ_V as constant

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